Enrichments over symmetric Picard categories

Vincent Schmitt
email: vrrschmitt@yahoo.fr

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Abstract

Categorical rings were introduced in [JiPi07], which we call 2-rings. In these notes we present basic definitions and results regarding 2-modules. This is work in progress.

1 Introduction

Categorical rings were introduced by M.Jibladze and T.Pirashvili in [JiPi07]. We call those 2-rings. The present set of notes contains basic results about 2-modules and this work is in progress. Section 2 contains technical preliminaries, namely reminders on symmetric Picard categories as well convenient references to previous works. In particular the developments in [Sch08] for symmetric monoidal categories transpose well to symmetric Picard categories and a suitable tensor product can be defined for the latter. Section 3 and 4 treat enrichments over symmetric Picard categories. Those were introduced in [Dup08] and defined by means of multilinear maps. We also define them using the tensor product. Section 5 contains expected examples of 2-rings and 2-enrichments. Section 6 contains basic results regarding categories of $\mathcal{A}$-modules for a 2-ring $\mathcal{A}$. In particular we show that $\mathcal{A}$-modules are particular algebras for the endo-2-functor $\mathcal{A} \otimes -$ of the 2-category of symmetric Picard categories. A large appendix contains the more technical developments.

2 Preliminaries

A categorical group structure $(\mathcal{A}, j)$ consists of a monoidal category $\mathcal{A}$ and an assignment for every object a of $\mathcal{A}$ of an object $a^\bullet$ with an isomorphism $j_a : \mathcal{I} \to a^\bullet \otimes a$, ($a^\bullet$ is an *inverse* of a). We are concerned in this paper with *symmetric Picard categories* which are the categorical groups $(\mathcal{A}, j)$ for which $\mathcal{A}$ has a symmetric monoidal structure and its underlying category is a groupoid. SPC denotes the 2-category with objects symmetric Picard categories, arrows symmetric monoidal functors and 2-cells monoidal natural transformations. There is a forgetful 2-functor $SPC \to SMC$ forgetting the group structure where SMC denotes the 2-category with objects symmetric monoidal categories, arrows symmetric monoidal functors, and monoidal natural transformations as 2-cells. The 2-categorical properties of SPC are similar to those of SMC, the latter 2-category has been studied in different works in particular in [HyPo02] and in [Sch08]. We refer the reader to this last work for basic notations, and more elaborate results. In this first section, we describe briefly and compare the important properties of SMC and SPC.

The 2-category SMC admits an internal hom and the same holds for SPC which hom is inherited from SMC. The following was mentioned in [Dup08] with a rather concise explanation.

Lemma 2.1 *Given any two objects in $\mathcal{A}$ and $\mathcal{B}$ in SMC with $\mathcal{B}$ being a symmetric Picard category, the internal hom $[\mathcal{A}, \mathcal{B}]$ in SMC admits a symmetric Picard structure given pointwise by that of $\mathcal{B}$.*

We present a proof that relies on coherence results for categorical groups from [Lap83]. (This work also contains references to earlier works on the topic.) Let us recall these. For a categorical group $\mathcal{A}$ with family of isomorphisms $j_a : I \to a^\bullet \otimes a$ for each object a there is a unique way of extending the assignment $a \mapsto a^\bullet$ into a functor $\mathcal{A}^{op} \to \mathcal{A}$ that makes the j_a natural in a. It is an equivalence. We will write it $(-)^\bullet$ and write $f^\bullet : b^\bullet \to a^\bullet$ for the image of any arrow $f : a \to b$ by this functor. There is a coherence theorem stating that any pair of a and b of objects of $\mathcal{A}$ there is at most one "canonical" arrow $a \to b$, those canonical arrows being the ones generated in an expected way from the canonical arrows from the monoidal structures and the j_a's. Eventually Laplaza's paper also provides a combinatorial description of free categorical groups.

Let us recall also the following known facts for any symmetric Picard categories $\mathcal{A}$ and $\mathcal{B}$. For any symmetric monoidal functor $(F, F^2, F^0) : \mathcal{A} \to \mathcal{B}$ the component F^0 is determined by F^2. Actually a monoidal structure on a functor $F : \mathcal{A} \to \mathcal{B}$ is given by a natural $F^2_{a,b} : Fa \otimes Fb \to F(a \otimes b)$ in $\mathcal{B}$ satisfying the only axiom that

$$
\begin{tikzcd}
Fa \otimes (Fb \otimes Fc) \arrow[r, dash, "\cong"] \arrow[d, "1 \otimes F^2_{b,c}"'] & (Fa \otimes Fb) \cong Fc \arrow[d, "F^2_{a,b} \otimes 1"] \\
Fa \otimes F(b \otimes c) \arrow[d, "F^2_{a,b\otimes c}"'] & F(a \otimes b) \otimes Fc \arrow[d, "F^2_{a\otimes b,1}"] \\
F(a \otimes (b \otimes c)) \arrow[r, dash, "F(\cong)"'] & F((a \otimes b) \otimes c)
\end{tikzcd}
$$

commutes for any objects a, b and c of $\mathcal{A}$. Also any natural transformation $\sigma : F \to G : \mathcal{A} \to \mathcal{B}$ between monoidal functors is monoidal if it satisfies the only axiom that the diagram in $\mathcal{B}$

$$
\begin{tikzcd}
Fa \otimes Fb \arrow[r, "F^2_{a,b}"] \arrow[d, "\sigma_a \otimes \sigma_b"'] & F(a \otimes b) \arrow[d, "\sigma_{a\otimes b}"] \\
Ga \otimes Gb \arrow[r, "G^2_{a,b}"'] & G(a \otimes b)
\end{tikzcd}
$$

commutes for any objects a,b of $\mathcal{A}$.

Let us consider a symmetric Picard category $(\mathcal{A}, j)$. Since $\mathcal{A}$ is a groupoid, one has a functor $\mathcal{A}^{op} \to \mathcal{A}$ which is the identity on objects and sends arrows to their inverses. The functor $inv : \mathcal{A} \to \mathcal{A}$ is obtained by composing the previous functors and $(-)^\bullet$ above. Let us denote by ! the unique canonical arrow between two objects of $\mathcal{A}$, when it exists.

Lemma 2.2 *For any symmetric Picard category $(\mathcal{A}, j)$, the associated functor inv admits a symmetric monoidal structure where inv^2 has component $inv^2_{a,b} : a^\bullet \otimes b^\bullet \to (a \otimes b)^\bullet$ in any (a, b) the composite $a^\bullet \otimes b^\bullet \xrightarrow{s} b^\bullet \otimes a^\bullet \xrightarrow{!} (a \otimes b)^\bullet$ (which is also $a^\bullet \otimes b^\bullet \xrightarrow{!} (b \otimes a)^\bullet \xrightarrow{s^\bullet} (a \otimes b)^\bullet$ according to Lemma 7.1 in Appendix).*

PROOF:See Appendix 7.2. ∎

2.3 *For any symmetric Picard category $(\mathcal{A}, j)$ one has a monoidal natural isomorphism $I \to inv\Box id : \mathcal{A} \to \mathcal{A}$ which component in any object a is $j_a : I \to a^\bullet \otimes a$.*

PROOF:See Appendix 7.4. ∎

Now given objects $\mathcal{A}$ and $\mathcal{B}$ in SMC with $(\mathcal{B}, j)$ symmetric Picard, a group structure is obtained on the hom $[\mathcal{A}, \mathcal{B}]$ in SMC, which is a groupoid, as follows. The strict symmetric monoidal functor $[\mathcal{A}, -] : [\mathcal{B}, \mathcal{B}] \to [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{B}]]$ sends the monoidal transformation $j : I \to inv \Box id : \mathcal{B} \to \mathcal{B}$ of 2.3 to a monoidal transformation

$$I = [\mathcal{A}, I] \xrightarrow{[\mathcal{A}, j]} [\mathcal{A}, inv \Box id] = [\mathcal{A}, inv] \Box [\mathcal{A}, id] = [\mathcal{A}, inv] \Box id \ : [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{B}].$$

which we define as the j on $[\mathcal{A}, \mathcal{B}]$. This is to say that $F^\bullet$ for any symmetric monoidal F is the composite

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{inv} \mathcal{B} \, .$$

and the natural isomorphisms $j_F : I \cong F^\bullet \Box F$ are pointwise $(j_F)_a : I \to (Fa)^\bullet \otimes Fa$. Then for any monoidal $\sigma : F \to G : \mathcal{A} \to \mathcal{B}$ one has that $\sigma^\bullet : G^\bullet \to F^\bullet$ is pointwise $(\sigma_a)^\bullet : (Ga)^\bullet \to (Fa)^\bullet$. We will always consider that this is the chosen group structure on $[\mathcal{A}, \mathcal{B}]$ when considered as an object of SPC. This structure is determined by that of $\mathcal{B}$.

One has a notion of *strictness* for arrows in SPC, which is *different* from the notion of strictness in SMC. Let us consider any symmetric Picard categories $\mathcal{A}$ and $\mathcal{B}$. For a symmetric monoidal functor $F : \mathcal{A} \to \mathcal{B}$ one has the natural isomorphism

2.4

$$F(a^\bullet) \cong_a (Fa)^\bullet.$$

defined precisely in Appendix-7.5. A symmetric monoidal functor $F : \mathcal{A} \to \mathcal{B}$ is called a *strict* arrow in SPC when it preserves strictly the monoidal structure and moreover preserves strictly the isomorphisms j meaning that the natural isomorphism 2.4 is an identity or equivalently that for any object a in $\mathcal{A}$, F sends $a^\bullet$ to $(Fa)^\bullet$ and $j_a : I \to a^\bullet \otimes a$ to $j_{Fa} : I \to (Fa)^\bullet \otimes Fa$. We write $StrSPC$ for the sub-2-category of SPC with same objects, strict arrows and 2-cells inherited from SPC.

The results from [Sch08] regarding SMC transpose rather straightforwardly to SPC as follows.

Given an arbitrary symmetric Picard category $\mathcal{C}$, it happens that the isomorphism $D_{\mathcal{A},\mathcal{B},\mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \to [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ defined in chapter 6 is also a strict arrow in SPC. For any arrow $F : \mathcal{A} \to [\mathcal{B}, C]$ in SPC, its image F^* by D, called it "dual", is strict in SPC if and only for any object b of $\mathcal{B}$, the arrow $F^*(b) : \mathcal{A} \to \mathcal{C}$ is strict in SPC. For any objects $\mathcal{A}$,$\mathcal{B}$ and $\mathcal{C}$ in SPC, the arrow $[\mathcal{A}, -]_{\mathcal{B},\mathcal{C}} : [\mathcal{B}, \mathcal{C}] \to [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$ defined in chapter 8 is strict in SPC.

The hom 2-functor of SMC defined in chapter 9 induces by restriction a hom 2-functor $SPC^{op} \times SPC \to SPC$ for SPC. The statements regarding the 2-naturality of D (chapter 10) and the evaluation functors (chapter 11) still hold when replacing formally SMC by SPC. In particular the evaluation functors are strict arrows in SPC.

Similarly to the case of the 2-category SMC, one has a tensor product in SPC. For any symmetric Picard categories $\mathcal{A}$ and $\mathcal{B}$, their tensor $\mathcal{A} \otimes \mathcal{B}$ satisfies the universal property of the existence of a 2-natural isomorphism

2.5

$$SPC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \cong_{\mathcal{C}} StrSPC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

between 2-functors $StrSPC \to \mathbf{Cat}$ in the argument $\mathcal{C}$. Note that the 2-naturality in question involves only *strict* morphisms in SPC. We briefly sketch a description of the above tensor $\mathcal{A} \otimes \mathcal{B}$ by generator and relations. It is similar to that given with more details in [Sch08] for the tensor

in SMC.

We consider a graph $\mathcal{H}$ with vertices the terms of the free $\{I,(-)^\bullet,\otimes\}$-algebra over the set $Obj(\mathcal{A})\times Obj(\mathcal{B})$, i.e. they are words of the formal language containing all pairs (a,b) – which we write $a\otimes b$ – for objects a of $\mathcal{A}$ and b of $\mathcal{B}$, the one-symbol word I, the words $X^\bullet$ for any vertex X, and $X\otimes Y$ for any vertices X and Y. The set of edges of $\mathcal{H}$ consists of:
- The "canonical" edges for the symmetric monoidal structure which are the $ass_{X,Y,Z}: X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$, $r_X: X\otimes I\to X$, $l_X: I\otimes X\to X$, $s_{X,Y}: X\otimes Y\to Y\otimes X$ for all vertices X,Y,Z;
- Edges $j_X: I\to X^\bullet\otimes X$, one for each vertex X;
- Edges $\gamma_{a,a',b}:(a\otimes b)\otimes(a'\otimes b)\to(a\otimes a')\otimes b$ and $\delta_{a,b,b'}:(a\otimes b)\otimes(a\otimes b')\to a\otimes(b\otimes b')$ indexed by objects a,a' of $\mathcal{A}$ and b,b' of $\mathcal{B}$;
- Edges $a\otimes f: a\otimes b\to a\otimes b'$ indexed by objects a of $\mathcal{A}$ and arrows $f: b\to b'$ of $\mathcal{B}$;
- Edges $f\otimes b: a\otimes b\to a'\otimes b$ indexed by objects b of $\mathcal{B}$ and arrows $f: a\to a'$ of $\mathcal{A}$;
- Edges $X\otimes p: X\otimes Y\to X\otimes Z$ and $p\otimes X: Y\otimes X\to Z\otimes X$ for any vertex X and any edge $p: Y\to Z$;
with the convention that edges above with different names are different.

Let us consider $\mathcal{FG}(\mathcal{H})$ the free groupoid on $\mathcal{H}$, i.e its arrows are mere concatenations of edges of $\mathcal{H}$ and their formal inverses. For any vertex X, one has two graph endomorphisms of $\mathcal{H}$, namely $X\otimes -$ and $-\otimes X$ sending respectively an arbitrary edge $f: Y\to Z$ to $X\otimes Y\to X\otimes Z$, resp. $Y\otimes X\to Z\otimes X$. These two extend uniquely to endofunctors of $\mathcal{FG}(\mathcal{H})$ and we extend the notation $X\otimes f$ and $f\otimes X$ to denote the images of arrows of $\mathcal{FG}(\mathcal{H})$ by these functors.

The tensor $\mathcal{A}\otimes\mathcal{B}$ is the quotient of $\mathcal{FG}(\mathcal{H})$ by the congruence generated by the following relations $\sim$ on its arrows from 2.6 to 2.20 below.

For all edges $X\xrightarrow{t}Y$ and $Z\xrightarrow{s}W$ of $\mathcal{H}$,

2.6

$$\begin{tikzcd}
 & X\otimes W \arrow[dr, "t\otimes W"] & \\
X\otimes Z \arrow[ur, "X\otimes s"] \arrow[dr, "t\otimes Z"'] & \sim & Y\otimes W. \\
 & Y\otimes Z \arrow[ur, "Y\otimes s"'] &
\end{tikzcd}$$

2.7 *Relations giving the coherence conditions for ass, r, l and s in $\mathcal{A}\otimes\mathcal{B}$.*

These are the following.
- For any vertices X, Y, Z and T,

$$\begin{tikzcd}
X\otimes(Y\otimes(Z\otimes T)) \arrow[r, "ass"] \arrow[d, "1\otimes ass"'] & (X\otimes Y)\otimes(Z\otimes T) \arrow[r, "ass"] & ((X\otimes Y)\otimes Z)\otimes T \\
X\otimes((Y\otimes Z)\otimes T) \arrow[rr, "ass"'] & \sim & (X\otimes(Y\otimes Z))\otimes T. \arrow[u, "ass\otimes 1"']
\end{tikzcd}$$

- For any vertices X and Y,

$$\begin{tikzcd}
X\otimes(I\otimes Y) \arrow[rr, "ass"] \arrow[dr, "1\otimes l"'] & \sim & (X\otimes I)\otimes Y \arrow[dl, "r\otimes 1"] \\
 & X\otimes Y. &
\end{tikzcd}$$

- For any vertex X,

$$\begin{tikzcd} X \otimes I \arrow[rr, "s"] \arrow[dr, "r"'] & \sim & I \otimes X \arrow[dl, "l"] \\ & X. & \end{tikzcd}$$

- For any vertices X, Y and Z,

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "ass"] \arrow[d, "1\otimes s"'] & (X \otimes Y) \otimes Z \arrow[r, "s"] & Z \otimes (X \otimes Y) \arrow[d, "ass"] \\ X \otimes (Z \otimes Y) \arrow[r, "ass"'] & (X \otimes Z) \otimes Y \arrow[u, phantom, "\sim"] & (Z \otimes X) \otimes Y. \arrow[l, "s\otimes 1"] \end{tikzcd}$$

2.8 *Relations for the naturalities of ass, r, l, and s in* $\mathcal{A} \otimes \mathcal{B}$.

For instance, one has for any edge $f : X \to X'$ of $\mathcal{H}$, and any vertices Y and Z,

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "ass_{X,Y,Z}"] \arrow[d, "f\otimes 1"'] \arrow[dr, phantom, "\sim"] & (X \otimes Y) \otimes Z \arrow[d, "(f\otimes 1)\otimes 1"] \\ X' \otimes (Y \otimes Z) \arrow[r, "ass_{X',Y,Z}"'] & (X' \otimes Y) \otimes Z. \end{tikzcd}$$

We will not write here the other relations. There are two more for the naturalities of $ass_{X,Y,Z}$ in Y and Z, one for that of l_X in X, one for that of r_X in X and two for those of $s_{X,Y}$ in X and Y.

For any object a in $\mathcal{A}$ and any arrows $b \xrightarrow{f} b' \xrightarrow{g} b''$ in $\mathcal{B}$,

2.9

$$\begin{tikzcd} a \otimes b \arrow[rr, "a\otimes(g\circ f)"] \arrow[dr, "a\otimes f"'] & \sim & a \otimes b'' \\ & a \otimes b' \arrow[ur, "a\otimes g"'] & \end{tikzcd}$$

For any object b in $\mathcal{B}$ and any arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ in $\mathcal{A}$,

2.10

$$\begin{tikzcd} a \otimes b \arrow[rr, "(g\circ f)\otimes b"] \arrow[dr, "f\otimes b"'] & \sim & a'' \otimes b \\ & a' \otimes b \arrow[ur, "g\otimes b"'] & \end{tikzcd}$$

For any objects a in $\mathcal{A}$ and b in $\mathcal{B}$,

2.11 $a \otimes id_b \sim id_{a\otimes b}$

and

2.12 $id_a \otimes b \sim id_{a\otimes b}$.

where id_b, id_a and $id_{a\otimes b}$ above are the identities respectively at b in $\mathcal{B}$, at a in $\mathcal{A}$ and at $a \otimes b$ in $\mathcal{FG}(\mathcal{H})$.

For any arrows $f : a \to a'$ in $\mathcal{A}$ and $g : b \to b'$ in $\mathcal{B}$,

2.13

$$\begin{tikzcd} a\otimes b \arrow[r, "f\otimes b"] \arrow[d, "a\otimes g"'] \arrow[dr, phantom, "\sim"] & a'\otimes b \arrow[d, "a'\otimes g"] \\ a\otimes b' \arrow[r, "f\otimes b'"'] & a'\otimes b'. \end{tikzcd}$$

2.14 *Relations for the "naturalities" of $\gamma_{a,a',b}$ in a, a' and b and $\delta_{a,b,b'}$ in a, b and b'.*

For instance by the relations for the "naturality" of $\gamma_{a,a',b}$ in b it is meant that for any objects a, a' in $\mathcal{A}$ and any arrow $g : b \to b'$ in $\mathcal{B}$,

$$\begin{tikzcd} (a\otimes b)\otimes(a'\otimes b) \arrow[r, "\gamma_{a,a',b}"] \arrow[d, "(1\otimes g)\otimes(1\otimes g)"'] \arrow[dr, phantom, "\sim"] & (a\otimes a')\otimes b \arrow[d, "1\otimes g"] \\ (a\otimes b')\otimes(a'\otimes b') \arrow[r, "\gamma_{a,a',b'}"'] & (a\otimes a')\otimes b'. \end{tikzcd}$$

We will not write explicitly now the five other relations.

For any objects a in $\mathcal{A}$ and b, b', b'' in $\mathcal{B}$,

2.15

$$\begin{tikzcd} (a\otimes b)\otimes((a\otimes b')\otimes(a\otimes b'')) \arrow[r, "ass"] \arrow[d, "1\otimes\delta_{a,b',b''}"'] & ((a\otimes b)\otimes(a\otimes b'))\otimes(a\otimes b'') \arrow[d, "\delta_{a,b,b'}\otimes 1"] \\ (a\otimes b)\otimes(a\otimes(b'\otimes b'')) \arrow[d, "\delta_{a,b,b'\otimes b''}"'] \arrow[r, phantom, "\sim"] & (a\otimes(b\otimes b'))\otimes(a\otimes b'') \arrow[d, "\delta_{a,b\otimes b',b''}"] \\ a\otimes(b\otimes(b'\otimes b'')) \arrow[r, "1\otimes ass_{b,b',b''}"'] & a\otimes((b\otimes b')\otimes b''). \end{tikzcd}$$

For any objects a in $\mathcal{A}$ and b, b' in $\mathcal{B}$,

2.16

$$\begin{tikzcd} (a\otimes b)\otimes(a\otimes b') \arrow[r, "\delta_{a,b,b'}"] \arrow[d, "s_{a\otimes b,a\otimes b'}"'] \arrow[dr, phantom, "\sim"] & a\otimes(b\otimes b') \arrow[d, "1\otimes s_{b,b'}"] \\ (a\otimes b')\otimes(a\otimes b) \arrow[r, "\delta_{a,b',b}"'] & a\otimes(b'\otimes b). \end{tikzcd}$$

For any objects a, a', a'' in $\mathcal{A}$ and b in $\mathcal{B}$,

2.17

$$\begin{tikzcd} (a\otimes b)\otimes((a'\otimes b)\otimes(a''\otimes b)) \arrow[r, "ass"] \arrow[d, "1\otimes\gamma_{a',a'',b}"'] & ((a\otimes b)\otimes(a'\otimes b))\otimes(a''\otimes b) \arrow[d, "\gamma_{a,a',b}\otimes 1"] \\ (a\otimes b)\otimes((a'\otimes a'')\otimes b)) \arrow[d, "\gamma_{a,a'\otimes a'',b}"'] \arrow[r, phantom, "\sim"] & ((a\otimes a')\otimes b)\otimes(a''\otimes b) \arrow[d, "\gamma_{a\otimes a',a'',b}"] \\ (a\otimes(a'\otimes a''))\otimes b \arrow[r, "ass\otimes 1"'] & ((a\otimes a')\otimes a'')\otimes b. \end{tikzcd}$$

For any objects a, a' in $\mathcal{A}$ and b in $\mathcal{B}$,

2.18

$$\begin{tikzcd}
(a \otimes b) \otimes (a' \otimes b) \arrow[r, "\gamma_{a,a',b}"] \arrow[d, "s_{a\otimes b, a'\otimes b}"'] \arrow[dr, phantom, "\sim"] & (a \otimes a') \otimes b \arrow[d, "s_{a,a'}\otimes 1"] \\
(a' \otimes b) \otimes (a \otimes b) \arrow[r, "\gamma_{a',a,b}"'] & (a' \otimes a) \otimes b.
\end{tikzcd}$$

For any objects a,a' in $\mathcal{A}$ and b,b' in $\mathcal{B}$,

2.19

$$\begin{tikzcd}
((a \otimes b) \otimes (a \otimes b')) \otimes ((a' \otimes b) \otimes (a' \otimes b')) \arrow[r, dash] \arrow[d, "\delta_{a,b,b'}\otimes\delta_{a',b,b'}"'] \arrow[dr, phantom, "\sim"] & ((a \otimes b) \otimes (a' \otimes b)) \otimes ((a \otimes b') \otimes (a' \otimes b')) \arrow[d, "\gamma_{a,a',b}\otimes\gamma_{a,a',b'}"] \\
(a \otimes (b \otimes b')) \otimes (a' \otimes (b \otimes b')) \arrow[d, "\gamma_{a,a',b\otimes b'}"'] & ((a \otimes a') \otimes b) \otimes ((a \otimes a') \otimes b') \arrow[dl, "\delta_{a\otimes a',b,b'}"] \\
(a \otimes a') \otimes (b \otimes b'). &
\end{tikzcd}$$

where the top arrow is the concatenation

$$\xrightarrow{ass_{X\otimes Y,Z,T}} \xrightarrow{ass^{-1}_{X,Y,Z}\otimes 1} \xrightarrow{(1\otimes s_{Y,Z})\otimes T} \xrightarrow{ass_{X,Z,Y}\otimes T} \xrightarrow{ass_{X\otimes Z,Y,T}{}^{-1}}$$

with the ass^{-1} being the formal inverses of edges ass and X, Y, Z and T standing respectively for $a \otimes b$, $a \otimes b'$, $a' \otimes b$ and $a' \otimes b'$.

2.20 *Expansions of all relations above by iterations of $X \otimes -$ and $- \otimes X$ for all vertices X.*

Which means precisely that the set of relations $\sim$ is the smallest set of relations on arrows of $\mathcal{FG}(\mathcal{H})$ containing the previous relations (2.6 to 2.19) and satisfying the closure properties that for any relation $f \sim g : Y \to Z$ that it contains and any vertex X, it contains also the relations

$$X \otimes f \sim X \otimes g : X \otimes Y \to X \otimes Z$$

and

$$f \otimes X \sim g \otimes X : Y \otimes X \to Z \otimes X.$$

The proofs that the above category $\mathcal{A} \otimes \mathcal{B}$ is a well defined symmetric Picard category and that it satisfies the universal property 2.5, are similar to those in [Sch08] for the well definition and universal property of the tensor product in SMC. We will therefore not replicate them. For any objects $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in SPC, one obtains an adjunction

$$En \dashv Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \to [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

in the 2-category SPC (in this case it is an equivalence) with $Rn \circ En = 1$ and where the arrows $Rn_{\mathcal{A},\mathcal{B},\mathcal{C}}$ are strict. The isomorphism 2.5 becomes 2-natural in $\mathcal{A}$ and $\mathcal{B}$ for a unique tensor 2-functor $SPC \times SPC \to SPC$.

There exists a free symmetric Picard category on one generator, which we shall write I, that differs obviously from the "unit" for SMC written I and defined in [Sch08]-chapter 18, but that has a very similar presentation by generators and relation. The only differences are the following. Its set objects is now the free $\{I, \otimes, (-)^\bullet\}$-algebra over one generator $\star$. It is a quotient of free *groupoid* generated by the graph containing the usual canonical edges for the symmetric monoidal structures

(the $ass_{X,Y,Z}$, r_X, l_X, $s_{X,Y}$) and moreover containing one edge $j_X : I \to X^\bullet \otimes X$ for each object X. The relations on this free groupoid defining I are just those for the naturalities of the collections ass, r, l and s, those expressing the coherence axioms for the symmetric monoidal structure, and relations expressing the bifunctoriality of $- \otimes - : I \times I \to I$. The universal property defining I is that for any symmetric Picard category $\mathcal{A}$, there exits a unique strict arrow $v : I \to [\mathcal{A}, \mathcal{A}]$ such that $v(\star)$ is the identity arrow at $\mathcal{A} \to \mathcal{A}$ with its strict structure in SPC. One obtains with similar proofs, similar results. Namely:

2.21 *For any symmetric Picard category $\mathcal{A}$, the dual $v^* : \mathcal{A} \to [\mathcal{I}, \mathcal{A}]$ of $v : \mathcal{I} \to [\mathcal{A}, \mathcal{A}]$ has right adjoint in SPC the evaluation at $\star$ functor $ev_\star : [\mathcal{I}, \mathcal{A}] \to \mathcal{A}$.*

From this one can exhibit a kind of "symmetric monoidal closed 2-structure" on SPC in the same way as done in [Sch08] chapters 19, 20 and 21 for SMC. Namely one can define the canonical arrows $A'_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \to \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$, $R'_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes I$, $L'_{\mathcal{A}} : \mathcal{A} \to I \otimes \mathcal{A}$ and $S_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ in this case with respective inverse equivalences $A_{\mathcal{A},\mathcal{B},\mathcal{C}}$, $R_{\mathcal{A}}$, $L_{\mathcal{A}}$ and $S_{\mathcal{B},\mathcal{A}}$ and satisfying the (strict) coherence axioms given in chapter 20. Eventually this 2-categorical structure induces a symmetric monoidal closed structure on $SPC/\sim$ where $\sim$ denotes the congruence generated by the 2-cells of SPC.

3 SPC-categories, SPC-functors and SPC-natural transformations

SPC-categories, SPC-functors and SPC-natural transformations have been considered by M.Dupont in his thesis [Dup08]. They are respectively bicategories with homs in SPC, and pseudo-functors and pseudo-natural transformations with linear components, i.e with arrows and 2-cells in the 2-category SPC rather than in **Cat**. As such they obviously form a 2-category with a forgetful 2-functor to the 2-category of bicategories, pseudo-functors and pseudo natural transformations.

We start by recalling these notions which are actually slight (enriched) variations of the usual notions of bicategories, pseudo-functors and pseudo-natural transformations.

By a n-linear natural transformation σ between n-linear maps F and G, written $\sigma : F \to G : \mathcal{A}_1 \times ... \times \mathcal{A}_n \to \mathcal{B}$ we will mean a 2-cell of $[\mathcal{A}_1, [\mathcal{A}_2, ..., [\mathcal{A}_n, \mathcal{B}]...]$. Multi-linear maps and multilinear natural transformation compose in an evident way, which can be justified by the 2-natural isomorphism $D_{\mathcal{A},\mathcal{B},\mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \cong [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ and the existence of a forgetful 2-functor $SPC \to \mathbf{Cat}$ (see [Sch08] chapter 6).

An *enrichment* $(\mathcal{A}, c, j, \alpha, \rho, \lambda)$, over SPC also named an *SPC-category* and which we might sometimes denote simply by $\mathcal{A}$, consists of the following data:
- A small set with elements x,y,z... called the *objects* of $\mathcal{A}$.
- A map $\mathcal{A}$ sending any pair x,y of objects to an object $\mathcal{A}(x, y)$ of SPC sometimes also written $\mathcal{A}_{x,y}$ for convenience and called the *hom* of x and y.
- A collection of bilinear maps $c_{x,y,z} : \mathcal{A}_{y,z} \times \mathcal{A}_{x,y} \to \mathcal{A}_{x,z}$, the *composition* maps indexed by objects x,y,z of $\mathcal{A}$ and we write $g \circ f$ for $c_{x,y,z}(g, f)$ for any objects g of $\mathcal{A}_{y,z}$ and f of $\mathcal{A}_{x,y}$ and $\tau * \sigma$ for $c_{x,y,z}(\tau, \sigma)$ for any arrows τ of $\mathcal{A}_{y,z}$ and σ of $\mathcal{A}_{x,y}$.
- A collection of objects 1_x of $\mathcal{A}_{x,x}$ indexed by objects x of $\mathcal{A}$;
- Collections of natural transformations $\alpha_{x,y,z,t}$, which are *trilinear*, indexed by objects x,y,z,t, and $\rho_{x,y}$ and $\lambda_{x,y}$ both *linear*, as follows

3.1

$$(\alpha_{x,y,z,t})_{h,g,f} : h \circ (g \circ f) \to h \circ (g \circ f)$$

which lies in $\mathcal{A}_{x,t}$ for objects h in $\mathcal{A}_{z,t}$, g in $\mathcal{A}_{y,z}$ and f in $\mathcal{A}_{x,y}$

3.2

$$(\rho_{x,y})_f : f \to f \circ 1_x$$

which lies in $\mathcal{A}_{x,y}$ for objects f in $\mathcal{A}_{x,y}$;

3.3

$$(\lambda_{x,y})_f : f \to 1_y \circ f$$

which lies in $\mathcal{A}_{x,y}$ for objects f in $\mathcal{A}_{x,y}$; and those are subjects to the coherence axioms 3.4 and 3.5 below.

3.4 *For any objects x, y, t and u of $\mathcal{A}$, any objects f of $\mathcal{A}_{x,y}$, g of $\mathcal{A}_{y,z}$, h of $\mathcal{A}_{z,t}$ and k of $\mathcal{A}_{t,u}$, the diagram in $\mathcal{A}_{x,u}$*

$$\begin{tikzcd}
k \circ (h \circ (g \circ f))) \arrow[rr, "(\alpha_{x,z,t,u})_{k,h,g\circ f}"] \arrow[d, "k*(\alpha_{x,y,z,t})_{h,g,f}"'] & & (k \circ h) \circ (g \circ f) \arrow[d, "(\alpha_{x,y,z,u})_{k\circ h,g,f}"] \\
k \circ ((h \circ g) \circ f) \arrow[r, "(\alpha_{x,y,t,u})_{k,h\circ g,f}"'] & (k \circ (h \circ g)) \circ f \arrow[r, "(\alpha_{y,z,t,u})_{k,h,g}*f"'] & ((k \circ h) \circ g) \circ f
\end{tikzcd}$$

commutes.

3.5 *For any objects x, y, z in $\mathcal{A}$ and any objects f of $\mathcal{A}(x,y)$ and g of $\mathcal{A}(y,z)$ the diagram in $\mathcal{A}(x,y)$*

$$\begin{tikzcd}
& g \circ f \arrow[dl, "g*(\lambda_{x,y})_f"'] \arrow[dr, "(\rho_{y,z})_g*f"] & \\
g \circ (1_y \circ f) \arrow[rr, "(\alpha_{x,y,y,z})_{g,1_y,f}"'] & & (g \circ 1_y) \circ f
\end{tikzcd}$$

commutes.

For any SPC-category $\mathcal{A}$, its underlying bicategory is denoted $\mathcal{A}^0$.

Given two arbitrary SPC-categories $\mathcal{A}$ and $\mathcal{B}$, a *SPC-functor* $F : \mathcal{A} \to \mathcal{B}$ consists of the following data.
- A map F sending objects of $\mathcal{A}$ to objects of $\mathcal{B}$;
- Arrows $F_{x,y} : \mathcal{A}_{x,y} \to \mathcal{B}_{Fx,Fy}$ in SPC for each pair of objects x,y in $\mathcal{A}$.
- A collection of *bilinear* natural transformations $F^2_{x,y,z}$ indexed by objects x,y,z of $\mathcal{A}$ with components

$$(F^2_{x,y,z})_{g,f} : F_{y,z}(g) \circ F_{x,y}(f) \to F_{x,z}(g \circ f)$$

in $\mathcal{B}_{Fx,Fz}$ for objects g in $\mathcal{A}_{y,z}$ and f in $\mathcal{A}_{x,y}$.
- A collection of arrows $F^0_x : 1_{Fx} \to F_{x,x}(1_x)$ in $\mathcal{B}_{Fx,Fx}$ indexed by objects x of $\mathcal{A}$.
Those are subjects to the coherence axioms 3.6, 3.7 and 3.8 below.

3.6 *For any objects x,y,z,t of $\mathcal{A}$, and any objects f of $\mathcal{A}_{x,y}$, g of $\mathcal{A}_{y,z}$ and h of $\mathcal{A}_{z,t}$ the diagram in $\mathcal{B}(Fx,Ft)$*

$$\begin{tikzcd}
F_{z,t}h \circ (F_{y,z}g \circ F_{x,y}f) \arrow[r, "(\alpha_{Fx,Fy,Fz,Ft})_{Fh,Fg,Ff}"] \arrow[d, "1*(F^2_{x,y,z})_{g,f}"'] & (F_{z,t}h \circ F_{y,z}g) \circ F_{x,y}f \arrow[d, "(F^2_{y,z,t})_{h,g}*1"] \\
F_{z,t}h \circ F_{x,z}(g \circ f) \arrow[d, "(F^2_{x,z,t})_{h,g\circ f}"'] & F_{y,t}(h \circ g) \circ F_{x,y}f \arrow[d, "(F^2_{x,y,t})_{h\circ g,f}"] \\
F_{x,t}(h \circ (g \circ f)) \arrow[r, "F_{x,t}((\alpha_{x,y,z,t})_{h,f,g})"'] & F_{x,t}((h \circ g) \circ f)
\end{tikzcd}$$

commutes.

3.7 *For any objects x,y of $\mathcal{A}$ and any object f of $\mathcal{A}_{x,y}$, the diagram in $\mathcal{B}_{Fx,Fy}$*

$$\begin{tikzcd}
Ff \arrow[r, "(\rho_{Fx,Fy})_{Ff}"] \arrow[d, "F(\rho_{x,y}{}_f)"'] & Ff \circ 1_{Fx} \arrow[d, "1 * F^0_x"] \\
F(f \circ 1_x) & Ff \circ F(1_x) \arrow[l, "(F^2_{x,x,y})_{f,1_x}"]
\end{tikzcd}$$

commutes.

3.8 *For any objects x,y of $\mathcal{A}$ and any object f of $\mathcal{A}_{x,y}$, the diagram in $\mathcal{B}_{Fx,Fy}$*

$$\begin{tikzcd}
Ff \arrow[r, "(\lambda_{Fx,Fy})_{Ff}"] \arrow[d, "F(\lambda_{x,y}{}_f)"'] & 1_{Fy} \circ Ff \arrow[d, "F^0_y * 1"] \\
F(1_y \circ f) & F(1_y) \circ Ff \arrow[l, "(F^2_{x,y,y})_{1_y,f}"]
\end{tikzcd}$$

commutes.

Given two SPC-functors $F, G : \mathcal{A} \to \mathcal{B}$ a *SPC-natural transformation* (σ, κ) consists in a family of arrows σ_x of $\mathcal{B}_{Fx,Gx}$ indexed by objects x of $\mathcal{A}$ together with a collection of linear natural transformations $\kappa_{x,y}$ indexed by objects x and y as follows

$$(\kappa_{x,y})_f : Gf \circ \sigma_x \to \sigma_y \circ Ff$$

lies in $\mathcal{B}_{Fx,Gy}$ for objects f of $\mathcal{A}_{x,y}$, and these satisfy the coherence axioms 3.9 and 3.10 below.

3.9 *For any objects f in $\mathcal{A}_{x,y}$ and g in $\mathcal{A}_{y,z}$, the diagram in $\mathcal{B}_{Fx,Gz}$*

$$\begin{tikzcd}
(Gg \circ Gf) \circ \sigma_x \arrow[r, "(G^2_{x,y,z})_{g,f} * \sigma_x"] & G(g \circ f) \circ \sigma_x \arrow[r, "(\kappa_{x,z})_{g \circ f}"] & \sigma_z \circ F(g \circ f) \\
Gg \circ (Gf \circ \sigma_x) \arrow[u, "(\alpha_{Fx,Gx,Gy,Gz})_{Gg,Gf,\sigma_x}"] \arrow[d, "1 * (\kappa_{x,y})_f"'] & & \sigma_z \circ (Fg \circ Ff) \arrow[u, "1 * (F^2_{x,y,z})_{g,f}"'] \arrow[d, "(\alpha_{Fx,Fx,Fz,Gz})_{\sigma_z,Fg,Ff}"] \\
Gg \circ (\sigma_y \circ Ff) \arrow[r, "(\alpha_{Fx,Fy,Gy,Gz})_{Gg,\sigma_y,Ff}"'] & (Gg \circ \sigma_y) \circ Ff \arrow[r, "(\kappa_{y,z})_g * 1"'] & (\sigma_z \circ Fg) \circ Ff
\end{tikzcd}$$

commutes.

3.10 *For any object x of $\mathcal{A}$, the diagram in $\mathcal{B}_{Fx,Gx}$*

$$\begin{tikzcd}
& \sigma_x \arrow[dl, "(\rho_{Fx,Gx})_{\sigma_x}"'] \arrow[dr, "(\lambda_{Fx,Gx})_{\sigma_x}"] & \\
\sigma_x \circ 1_{Fx} \arrow[d, "\sigma_x * F^0_x"'] & & 1_{Gx} \circ \sigma_x \arrow[d, "G^0_x * \sigma_x"] \\
\sigma_x \circ F(1_x) & & G(1_x) \circ \sigma_x \arrow[ll, "(\kappa_{x,x})_{1_x}"]
\end{tikzcd}$$

commutes.

We want to give alternative definitions of SPC-categories, SPC-functors and SPC-natural transformations by means of commuting diagrams in SPC in a first instance without using the tensor. They are obtained by replacing multilinear maps and multilinear natural transformations from the previous definition by corresponding arrows and 2-cells in SPC. This yields the following.

One can define a SPC-category as a collection of objects $\mathcal{A}$, with homs $\mathcal{A}_{x,y}$ in SPC as before, with collections of arrows
- $\mathcal{A}(x,-)_{y,z} : \mathcal{A}_{y,z} \to [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]$ in SPC, indexed by objects x, y, z of $\mathcal{A}$ with dual $\mathcal{A}_{-,y} : \mathcal{A}_{x,y} \to [\mathcal{A}_{y,z}, \mathcal{A}_{x,z}]$ written $\mathcal{A}(-,y)$;
- $u_x : \mathcal{I} \to \mathcal{A}_{x,x}$ indexed by x, which are strict;
and collections of 2-cells:
- $\alpha'_{x,y,z,t}$ in SPC, indexed by objects x, y, z and t of SPC as follows

3.11

$$\begin{tikzcd}
 & \mathcal{A}_{z,t} \arrow[dl, "\mathcal{A}(x,-)"'] \arrow[dr, "\mathcal{A}(y,-)"] & \\
{[\mathcal{A}_{x,z}, \mathcal{A}_{x,t}]} \arrow[d, "{[\mathcal{A}_{x,y},-]}"'] & & {[\mathcal{A}_{y,z}, \mathcal{A}_{y,t}]} \arrow[d, "{[1,\mathcal{A}(x,-)]}"] \\
{[[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,t}]]} \arrow[rr, "{[\mathcal{A}(x,-),1]}"'] \arrow[urr, Rightarrow, "\alpha'"] & & {[\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,t}]]}
\end{tikzcd}$$

- $\rho'_{x,y}$ and $\lambda'_{x,y}$ indexed by objects x and y respectively as follows

3.12 $\rho'_{x,y}$:

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, no head, "id"'] \arrow[r, phantom, shift right=6ex, "\Longrightarrow"] & {[\mathcal{A}_{x,x}, \mathcal{A}_{x,y}]} \arrow[d, "{[u_x,1]}"] \\
\mathcal{A}_{x,y} & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[l, "ev_\star"]
\end{tikzcd}$$

and

3.13 $\lambda'_{x,y}$:

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "\mathcal{A}(-,y)"] \arrow[d, no head, "id"'] \arrow[r, phantom, shift right=6ex, "\Longrightarrow"] & {[\mathcal{A}_{y,y}, \mathcal{A}_{x,y}]} \arrow[d, "{[u_y,1]}"] \\
\mathcal{A}_{x,y} & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[l, "ev_\star"]
\end{tikzcd}$$

those satisfying coherence axioms 3.14 and 3.15 below

3.14 *The 2-cells in SPC*

$$\begin{tikzcd}
\mathcal{A}_{t,u} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{x,t},\mathcal{A}_{x,u}]} \arrow[r, "{[\mathcal{A}_{x,z},-]}"] \arrow[d, phantom, "\Downarrow \alpha'_{x,z,t,u}"] & {[[\mathcal{A}_{x,z},\mathcal{A}_{x,t}],[\mathcal{A}_{x,z},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}(x,-),1]}"] & {[\mathcal{A}_{z,t},[\mathcal{A}_{x,z},\mathcal{A}_{x,u}]]} \arrow[r, "{[1,[\mathcal{A}_{x,y},-]]}"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{z,t},[[\mathcal{A}_{x,y},\mathcal{A}_{x,z}],[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[1,[\mathcal{A}(x,-),1]]}"] \arrow[d, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(z,-)"] \arrow[d, dash, "id"'] & \arrow[d, phantom, "="] & {[\mathcal{A}_{z,t},\mathcal{A}_{z,u}]} \arrow[r, "{[1,\mathcal{A}(x,-)]}"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{z,t},[\mathcal{A}_{x,z},\mathcal{A}_{x,u}]]} \arrow[r, "{[1,[\mathcal{A}_{x,y},-]]}"] \arrow[d, phantom, "{[1,\alpha'_{x,y,z,u}]}\Downarrow"] & {[\mathcal{A}_{z,t},[[\mathcal{A}_{x,y},\mathcal{A}_{x,z}],[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[1,[\mathcal{A}(x,-),1]]}"] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(z,-)"'] & & {[\mathcal{A}_{z,t},\mathcal{A}_{z,u}]} \arrow[rr, "{[1,\mathcal{A}(y,-)]}"'] & & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},\mathcal{A}_{y,u}]]} \arrow[r, "{[1,[1,\mathcal{A}(x,-)]]}"'] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_{t,u} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{x,t},\mathcal{A}_{x,u}]} \arrow[r, "{[\mathcal{A}_{x,y},-]}"] \arrow[d, phantom, "="] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,t}],[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] & {[[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[[\mathcal{A}(x,-),1],1]}"] \arrow[d, dash, "id"'] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,z}],[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[[\mathcal{A}_{x,y},-],1]}"] \arrow[d, phantom, "{[\alpha'_{x,y,z,t},1]}\Downarrow"] & {[[\mathcal{A}_{x,z},\mathcal{A}_{x,t}],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, "{[\mathcal{A}(x,-),1]}"] \\
\mathcal{A}_{t,u} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{x,t},\mathcal{A}_{x,u}]} \arrow[r, "{[\mathcal{A}_{x,y},-]}"] \arrow[d, phantom, "="] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,t}],[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] \arrow[d, dash, "id"'] & {[[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[[1,\mathcal{A}(x,-)],1]}"] \arrow[d, phantom, "="] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[\mathcal{A}(y,-),1]}"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{x,t},\mathcal{A}_{x,u}]} \arrow[r, "{[\mathcal{A}_{x,y},-]}"] \arrow[d, phantom, "\Downarrow \alpha'_{x,y,t,u}"] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,t}],[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}(x,-),1]}"] & {[\mathcal{A}_{y,t},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] \arrow[d, dash, "id"'] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[\mathcal{A}(y,-),1]}"] \arrow[d, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(y,-)"] \arrow[d, dash, "id"'] & \arrow[d, phantom, "="] & {[\mathcal{A}_{y,t},\mathcal{A}_{y,u}]} \arrow[r, "{[1,\mathcal{A}(x,-)]}"] \arrow[d, dash, "id"'] & {[\mathcal{A}_{y,t},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] \arrow[d, phantom, "="] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[\mathcal{A}(y,-),1]}"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(y,-)"] \arrow[d, dash, "id"'] & \arrow[d, phantom, "="] & {[\mathcal{A}_{y,t},\mathcal{A}_{y,u}]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},\mathcal{A}_{y,u}]]} \arrow[r, "{[1,[1,\mathcal{A}(x,-)]]}"] \arrow[d, dash, "id"'] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[r, "{[\mathcal{A}(y,-),1]}"] \arrow[d, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(y,-)"] \arrow[d, dash, "id"'] & & {[\mathcal{A}_{y,t},\mathcal{A}_{y,u}]} \arrow[r, "{[\mathcal{A}_{y,z},-]}"] \arrow[d, phantom, "\Downarrow \alpha'_{y,z,t,u}"] & {[[\mathcal{A}_{y,z},\mathcal{A}_{y,t}],[\mathcal{A}_{y,z},\mathcal{A}_{y,u}]]} \arrow[r, "{[\mathcal{A}(y,-),1]}"] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},\mathcal{A}_{y,u}]]} \arrow[r, "{[1,[1,\mathcal{A}(x,-)]]}"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "="] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]} \arrow[d, dash, "id"] \\
\mathcal{A}_{t,u} \arrow[rr, "\mathcal{A}(z,-)"'] & & {[\mathcal{A}_{z,t},\mathcal{A}_{z,u}]} \arrow[rr, "{[1,\mathcal{A}(y,-)]}"'] & & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},\mathcal{A}_{y,u}]]} \arrow[r, "{[1,[1,\mathcal{A}(x,-)]]}"'] & {[\mathcal{A}_{z,t},[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,u}]]]}
\end{tikzcd}$$

are equal.

3.15 *The 2-cell*

$$\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[rr, dash, "id"] \arrow[d, "\mathcal{A}(y,-)"'] & \arrow[d, phantom, "\Downarrow \rho'"] & \mathcal{A}_{y,z} \arrow[r, "\mathcal{A}(x,-)"] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \\
{[\mathcal{A}_{y,y},\mathcal{A}_{y,z}]} \arrow[rr, "{[u_y,1]}"'] & & {[\mathcal{I},\mathcal{A}_{y,z}]} \arrow[u, "ev_\star"'] &
\end{tikzcd}$$

is equal to

$$
\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, dash, "id"] \arrow[d, "\mathcal{A}(x,-)"'] \arrow[dr, phantom, "="] & \mathcal{A}_{y,z} \arrow[r, dash, "id"] \arrow[d, "\mathcal{A}(x,-)"] \arrow[dr, phantom, "="] & \mathcal{A}_{y,z} \arrow[r, dash, "id"] \arrow[d, "\mathcal{A}(x,-)"] & \mathcal{A}_{y,z} \arrow[r, dash, "id"] \arrow[d, "\mathcal{A}(x,-)"] \arrow[dr, phantom, "="] & \mathcal{A}_{y,z} \arrow[d, "\mathcal{A}(x,-)"] \\
{[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"] \arrow[dddd, dash, "id"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"] \arrow[d, "{[ev_\star,1]}"] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[d, "{[\mathcal{A}_{x,y},-]}"'] \arrow[r, phantom, "\overset{\alpha'_{x,y,y,z}}{\Longrightarrow}"] & {[\mathcal{A}_{y,y},\mathcal{A}_{y,z}]} \arrow[r, dash, "id"] \arrow[dd, "{[1,\mathcal{A}(x,-)]}"] & {[\mathcal{A}_{y,y},\mathcal{A}_{y,z}]} \arrow[d, "{[u_y,1]}"] \\
 & {[[\mathcal{I},\mathcal{A}_{x,y}],\mathcal{A}_{x,z}]} \arrow[d, "{[[u_y,1],1]}"] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,y}],[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]]} \arrow[d, "{[\mathcal{A}(x,-),1]}"] & & {[\mathcal{I},\mathcal{A}_{y,z}]} \arrow[d, "ev_\star"] \\
\arrow[r, phantom, "\overset{[\lambda'_{x,y},1]}{\Longrightarrow}"] & {[[\mathcal{A}_{y,y},\mathcal{A}_{x,y}],\mathcal{A}_{x,z}]} \arrow[dd, "{[\mathcal{A}(-,y),1]}"'] \arrow[r, phantom, "=_{(I)}"] & {[\mathcal{A}_{y,y},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]]} \arrow[r, dash, "id"] \arrow[d, "{[u_y,1]}"] & {[\mathcal{A}_{y,y},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]]} \arrow[d, "{[u_y,1]}"] \arrow[r, phantom, "=_{(II)}"] & \mathcal{A}_{y,z} \arrow[dd, "\mathcal{A}(x,-)"] \\
 & & {[\mathcal{I},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]]} \arrow[d, "ev_\star"] \arrow[r, phantom, "="] & {[\mathcal{I},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]]} \arrow[d, "ev_\star"] & \\
{[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, dash, "id"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]}.
\end{tikzcd}
$$

Note: Equality (I) in the first of the pastings above is established in 7 in Appendix. The equality (II) results straightforwardly from Lemma [Sch08]-11.2.

Let us justify the equivalence with the previous definition of SPC-category. For objects x, y and z the arrows of SPC

$$\mathcal{A}(x,-) : \mathcal{A}(y,z) \to [\mathcal{A}(x,y), \mathcal{A}(x,z)]$$

correspond to the bilinear

$$c_{x,y,z} : \mathcal{A}(y,z) \times \mathcal{A}(x,y) \to \mathcal{A}(x,z)$$

and for objects x, the objects 1_x in $\mathcal{A}_{x,x}$ correspond to strict the strict arrows $u_x : \mathcal{I} \to \mathcal{A}_{x,x}$. The trilinear 2-cells $\alpha_{x,y,z,t}$ correspond to the 2-cells $\alpha'_{x,y,z,t}$ and the linear natural transformations $\rho_{x,y}$ and $\lambda_{x,y}$ correspond respectively to 2-cells $\rho'_{x,y}$ and $\lambda'_{x,y}$ in SPC.
For data as above, since the forgetful functors $SPC(X,Y) \to \mathbf{Cat}(X,Y)$ are faithful, the equalities of 2-cells of Axiom 3.14 given below are equivalent to the equality of natural transformations of Axioms 3.4, and similarly Axiom 3.15 given below and 3.5 are equivalent.

Given two SPC-categories, a SPC-functor $\mathcal{A} \to \mathcal{B}$ consists of a map F sending objects of $\mathcal{A}$ to objects of $\mathcal{B}$, with a collection of arrows in SPC $F_{x,y} : \mathcal{A}_{x,y} \to \mathcal{B}_{Fx,Fy}$ indexed by objects x and y of $\mathcal{A}$ and collections of 2-cells in SPC:
- ${F'}^2_{x,y,z}$ indexed by objects x, y, z and as follows

3.16

$$
\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, "F_{y,z}"] \arrow[d, "\mathcal{A}(x,-)"'] & \mathcal{B}_{Fy,Fz} \arrow[r, "\mathcal{B}(Fx,-)"] \arrow[d, phantom, "\overset{{F'}^2_{x,y,z}}{\Longleftarrow}"] & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}]} \arrow[d, "{[F_{x,y},1]}"] \\
{[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[rr, "{[1,F_{x,z}]}"'] & & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Fz}]}.
\end{tikzcd}
$$

- F^0_x indexed by objects x as follows

3.17

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_x"] \arrow[dr, "u_{Fx}"'] \arrow[dr, phantom, "\Uparrow F^0_x", near start, shift left=4] & \mathcal{A}_{x,x} \arrow[d, "F_{x,x}"] \\
& \mathcal{B}_{Fx,Fx}
\end{tikzcd}$$

Those satisfy the coherence conditions 3.18, 3.19 and 3.20 below.

3.18 *The 2-cells in SPC*

$$\begin{tikzcd}
& & {[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}]} \arrow[r, "{[\mathcal{B}_{Fx,Fy},-]}"] \arrow[d, Rightarrow, "\alpha'_{Fx,Fy,Fz,Ft}"] & {[[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]]} \arrow[d, "{[\mathcal{B}(Fx,-),1]}"] & \\
\mathcal{A}_{z,t} \arrow[r, "F_{z,t}"] \arrow[d, "{\mathcal{A}(y,-)}"'] & \mathcal{B}_{Fz,Ft} \arrow[ur, "{\mathcal{B}(Fx,-)}"] \arrow[r, "{\mathcal{B}(Fy,-)}"] & {[\mathcal{B}_{Fy,Fz},\mathcal{B}_{Fy,Ft}]} \arrow[r, "{[1,\mathcal{B}(Fx,-)]}"] \arrow[d, "{[F_{y,z},1]}"] \arrow[dll, Rightarrow, "{F'^2_{y,z,t}}"'] \arrow[dr, phantom, "="] & {[\mathcal{B}_{Fy,Fz},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]]} \arrow[d, "{[F_{y,z},1]}"] & \\
{[\mathcal{A}_{y,z},\mathcal{A}_{y,t}]} \arrow[rr, "{[1,F_{y,t}]}"] \arrow[drr, "{[1,\mathcal{B}(Fx,-)]}"'] & & {[\mathcal{A}_{y,z},\mathcal{B}_{Fy,Ft}]} \arrow[r, "{[1,\mathcal{B}(Fx,-)]}"] \arrow[d, Rightarrow, "{[1,F'^2_{x,y,t}]}"] & {[\mathcal{A}_{y,z},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]]} \arrow[r, "{[1,[F_{x,y},1]]}"] & {[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \\
& & {[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]]} \arrow[urr, "{[1,[1,F_{x,t}]]}"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{B}_{Fz,Ft} \arrow[r, "{\mathcal{B}(Fx,-)}"] & {[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}]} \arrow[r, "{[\mathcal{A}_{x,y},-]}"] \arrow[d, "{[F_{x,z},1]}"] \arrow[ddl, Rightarrow, "{F'^2_{x,z,t}}"'] \arrow[dr, phantom, "="] & {[[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Fz}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \arrow[r, "{[[F_{x,y},1],1]}"] \arrow[d, "{[[1,F_{x,z}],1]}"] & {[[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \arrow[r, "{[\mathcal{B}(Fx,-),1]}"] \arrow[d, phantom, "\Downarrow {[F'^2_{x,y,z},1]}"] & {[\mathcal{B}_{Fy,Fz},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \arrow[d, "{[F_{y,z},1]}"] \\
& {[\mathcal{A}_{y,z},\mathcal{B}_{Fx,Ft}]} \arrow[r, "{[\mathcal{A}_{x,y},-]}"] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,z}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \arrow[rr, "{[\mathcal{A}(x,-),1]}"'] \arrow[drr, phantom, "="] & & {[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]} \\
\mathcal{A}_{z,t} \arrow[uu, "F_{z,t}"] \arrow[r, "{\mathcal{A}(x,-)}"] \arrow[drr, "{\mathcal{A}(y,-)}"'] & {[\mathcal{A}_{x,z},\mathcal{A}_{x,t}]} \arrow[u, "{[1,F_{x,t}]}"] \arrow[ur, phantom, "="] \arrow[r, "{[\mathcal{A}_{x,y},-]}"] & {[[\mathcal{A}_{x,y},\mathcal{A}_{x,z}],[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]]} \arrow[u, "{[1,[1,F_{x,t}]]}"] \arrow[rr, "{[\mathcal{A}(x,-),1]}"] \arrow[d, Rightarrow, "\alpha'_{x,y,z,t}"] & & {[\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]]} \arrow[u, "{[1,[1,F_{x,t}]]}"'] \\
& & {[\mathcal{A}_{y,z},\mathcal{A}_{y,t}]} \arrow[urr, "{[1,\mathcal{A}(x,-)]}"'] & &
\end{tikzcd}$$

are equal.

3.19 *The 2-cells*

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[rr, "F_{x,y}"] \arrow[d, "\mathcal{A}(x,-)"'] & & \mathcal{B}_{Fx,Fy} \arrow[d, "\mathcal{B}(Fx,-)"] \arrow[dll, Rightarrow, "{F'}^2_{x,x,y}"'] \arrow[dddd, bend left=60, "1"] & \\
{[\mathcal{A}_{x,x}, \mathcal{A}_{x,y}]} \arrow[dr, "{[1,F_{x,y}]}"] \arrow[dd, "{[u_x,1]}"'] & & {[\mathcal{B}_{Fx,Fx}, \mathcal{B}_{Fx,Fy}]} \arrow[dl, "{[F_{x,x},1]}"'] \arrow[dd, "{[u_{Fx},1]}"] & \\
{} \arrow[dr, phantom, "="] & {[\mathcal{A}_{x,x}, \mathcal{B}_{Fx,Fy}]} \arrow[dr, "{[u_x,1]}"] & {} \arrow[l, Rightarrow, "{[F^0_x,1]}"'] & {} \arrow[l, Rightarrow, "\rho'_{Fx,Fy}"'] \\
{[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[rr, "{[1,F_{x,y}]}"] \arrow[d, "ev_\star"'] \arrow[drr, phantom, "="] & & {[\mathcal{I}, \mathcal{B}_{Fx,Fy}]} \arrow[d, "ev_\star"] & \\
\mathcal{A}_{x,y} \arrow[rr, "F_{x,y}"'] & & \mathcal{B}_{Fx,Fy} &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "id"] \arrow[d, "\mathcal{A}(x,-)"'] \arrow[dr, phantom, "\Downarrow \rho'_{x,y}"] & \mathcal{A}_{x,y} \arrow[r, "F_{x,y}"] & \mathcal{B}_{Fx,Fy} \\
{[\mathcal{A}_{x,x}, \mathcal{A}_{x,y}]} \arrow[r, "{[u_x,1]}"'] & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[u, "ev_\star"'] &
\end{tikzcd}
$$

are equal.

3.20 *The 2-cells in SPC*

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[rr, "F_{x,y}"] \arrow[d, "\mathcal{A}(-,y)"'] & & \mathcal{B}_{Fx,Fy} \arrow[d, "\mathcal{B}(-,Fy)"] \arrow[dll, Rightarrow, "({F'}^2_{x,y,y})^*"'] \arrow[dddd, bend left=60, "1"] & \\
{[\mathcal{A}_{y,y}, \mathcal{A}_{x,y}]} \arrow[dr, "{[1,F_{x,y}]}"] \arrow[dd, "{[u_y,1]}"'] & & {[\mathcal{B}_{Fy,Fy}, \mathcal{B}_{Fx,Fy}]} \arrow[dl, "{[F_{x,y},1]}"'] \arrow[dd, "{[u_{Fy},1]}"] & \\
{} \arrow[dr, phantom, "="] & {[\mathcal{A}_{y,y}, \mathcal{B}_{Fx,Fy}]} \arrow[dr, "{[u_y,1]}"] & {} \arrow[l, Rightarrow, "{[F^0_y,1]}"'] & {} \arrow[l, Rightarrow, "\lambda'_{Fx,Fy}"'] \\
{[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[rr, "{[1,F_{x,y}]}"] \arrow[d, "ev_\star"'] \arrow[drr, phantom, "="] & & {[\mathcal{I}, \mathcal{B}_{Fx,Fy}]} \arrow[d, "ev_\star"] & \\
\mathcal{A}_{x,y} \arrow[rr, "F_{x,y}"'] & & \mathcal{B}_{Fx,Fy} &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "id"] \arrow[d, "\mathcal{A}(-,y)"'] \arrow[dr, phantom, "\Downarrow \lambda'_{x,y}"] & \mathcal{A}_{x,y} \arrow[r, "F_{x,y}"] & \mathcal{B}_{Fx,Fy} \\
{[\mathcal{A}_{y,y}, \mathcal{A}_{x,y}]} \arrow[r, "{[u_y,1]}"'] & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[u, "ev_\star"'] &
\end{tikzcd}
$$

are equal.

One can also define the SPC-natural transformations, in a similar way. For this purpose, we need some notation.

Let $\mathcal{A}$ be an arbitrary SPC-category. To give a *strict* arrow $\mathcal{I} \to \mathcal{A}(x,y)$ is equivalent to give an arrow $x \to y$ of the underlying bicategory $\mathcal{A}^0$ and we might confuse the two. We therefore define for any object z of $\mathcal{A}$ and any arrow $f : x \to y$ of $\mathcal{A}^0$ the arrows

3.21 $\mathcal{A}(f,1)$ *as the composite arrow in SPC*

$$\mathcal{A}_{x,y} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y},\mathcal{A}_{x,z}] \xrightarrow{[F,1]} [\mathcal{I},\mathcal{A}_{x,z}] \xrightarrow{ev_\star} \mathcal{A}_{x,z}$$

and

3.22 $\mathcal{A}(1,f)$ *as*

$$\mathcal{A}_{z,x} \xrightarrow{\mathcal{A}(-,y)} [\mathcal{A}_{x,y},\mathcal{A}_{z,y}] \xrightarrow{[F,1]} [\mathcal{I},\mathcal{A}_{z,y}] \xrightarrow{ev_\star} \mathcal{A}_{z,y}.$$

Given any arrows $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$ in $\mathcal{A}^0$, we define the two cells

3.23 ${c^1}_{f,y,z}$

$$\begin{tikzcd}
\mathcal{A}_{z,t} \arrow[r, "\mathcal{A}(y,-)"] \arrow[d, "\mathcal{A}(x,-)"'] & {[\mathcal{A}_{y,z},\mathcal{A}_{y,t}]} \arrow[d, "{[1,\mathcal{A}(f,1)]}"] \\
{[\mathcal{A}_{x,z},\mathcal{A}_{x,t}]} \arrow[r, "{[\mathcal{A}(f,1),1]}"'] \arrow[ur, Rightarrow] & {[\mathcal{A}_{y,z},\mathcal{A}_{x,t}]}
\end{tikzcd}$$

3.24 ${c^2}_{x,g,t}$

$$\begin{tikzcd}
\mathcal{A}_{z,t} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, "\mathcal{A}(g,1)"'] & {[\mathcal{A}_{x,z},\mathcal{A}_{x,t}]} \arrow[d, "{[\mathcal{A}(1,g),1]}"] \arrow[dl, Rightarrow] \\
\mathcal{A}_{y,t} \arrow[r, "\mathcal{A}(x,-)"'] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]}
\end{tikzcd}$$

and eventually

3.25 ${c^3}_{x,y,h}$

$$\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, "\mathcal{A}(1,h)"] \arrow[d, "\mathcal{A}(x,-)"'] & \mathcal{A}_{y,t} \arrow[d, "\mathcal{A}(x,-)"] \\
{[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[r, "{[1,\mathcal{A}(1,h)]}"'] \arrow[ur, Rightarrow] & {[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]}
\end{tikzcd}$$

which are obtained from the trilinear natural transformation 3.1

$$(\alpha_{x,y,z,t})_{h,g,f} : h \circ (g \circ f) \to (h \circ g) \circ f$$

by fixing one of its argument. For c^1, c^2 and c^3 fix respectively f,g and h.

Formally ${c^1}_{f,z,t}$ is the composite

$$\mathcal{A}_{z,t} \xRightarrow{\alpha'_{x,y,z,t}} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]] \xrightarrow{[1,[f,1]]} [\mathcal{A}_{y,z},[\mathcal{I},\mathcal{A}_{x,t}]] \xrightarrow{[1,ev_\star]} [\mathcal{A}_{y,z},\mathcal{A}_{x,t}]$$

(see 7.23 in Appendix), ${c^2}_{x,g,t}$ it is the composite

$$\mathcal{A}_{z,t} \xrightarrow{\alpha'_{x,y,z,t}} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]] \xrightarrow{[g,1]} [\mathcal{I},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]] \xrightarrow{ev_\star} [\mathcal{A}_{x,y},\mathcal{A}_{x,t}]$$

and ${c^3}_{x,y,h}$ is the image by $ev_\star$ of the composite

$$\mathcal{I} \xrightarrow{h} \mathcal{A}_{z,t} \xRightarrow{\alpha'_{x,y,z,t}} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{A}_{x,t}]]$$

(see 7.26 in Appendix). By definition all 2-cells c^1, c^2 and c^3 are identities when $\mathcal{A}$ is strict.

One has also for any objects x and y of $\mathcal{A}$ the 2-cell in SPC

3.26 $r'_{x,y}$:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_x"] \arrow[dr, "v"'] \arrow[dr, phantom, shift left=3, "\Nearrow"] & \mathcal{A}_{x,x} \arrow[d, "\mathcal{A}(-,y)"] \\
& {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]}
\end{tikzcd} ,$$

which according to Remarks 7.10 is determined by its value in $\star$ which is the linear natural transformation

$$\rho_{x,y\,f} : f \circ 1_x \to f$$

of 3.2, and corresponds by the bijection 7.16/7.18 in Appendix to the 2-cell $\rho'_{x,y}$

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "\mathcal{A}(x,-)"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "\Rightarrow"] & {[\mathcal{A}_{x,x}, \mathcal{A}_{x,y}]} \arrow[d, "{[u_x,1]}"] \\
\mathcal{A}_{x,y} & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[l, "ev_\star"]
\end{tikzcd}.$$

Similarly for any objects x and y of $\mathcal{A}$ one has the 2-cell in SPC

3.27 $l'_{x,y}$:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_y"] \arrow[dr, "v"'] \arrow[dr, phantom, shift left=3, "\Nearrow"] & \mathcal{A}_{y,y} \arrow[d, "\mathcal{A}(x,-)"] \\
& {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]}
\end{tikzcd}$$

that corresponds to the linear natural transformation

$$(\lambda_{x,y})_f : 1_y \circ f \to f$$

of 3.3 and which corresponds by the bijection 7.16/7.18 in Appendix to the 2-cell $\lambda'_{x,y}$

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "\mathcal{A}(-,y)"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "\Rightarrow"] & {[\mathcal{A}_{y,y}, \mathcal{A}_{x,y}]} \arrow[d, "{[u_y,1]}"] \\
\mathcal{A}_{x,y} & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[l, "ev_\star"]
\end{tikzcd}.$$

Given any strict arrow $\tilde{f} : \mathcal{I} \to \mathcal{A}_{x,y}$, with corresponding arrow $f : x \to y$ in $\mathcal{A}^0$, one has the 2-cell in SPC

3.28 $u^1{}_f$:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_y"] \arrow[dr, "\tilde{f}"'] \arrow[r, phantom, shift right=4ex, "\Uparrow"] & \mathcal{A}_{y,y} \arrow[d, "\mathcal{A}(f,1)"] \\
 & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]}
\end{tikzcd}$$

which corresponds to the *2-cell* $\rho_f : f \to 1_y \circ f : x \to y$ of $\mathcal{A}^0$. It is the pasting

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_y"] \arrow[rr, bend right=50, "v"'] \arrow[rrrr, bend right=70, "\tilde{f}"'] & \mathcal{A}_{y,y} \arrow[r, "\mathcal{A}(x,-)"'] \arrow[rrr, bend left=40, "\mathcal{A}(f,1)"] \arrow[d, phantom, "\Uparrow l'"] & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]} \arrow[r, "{[\tilde{f},1]}"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[r, "ev_\star"] & \mathcal{A}_{x,y} \\
 & {} & {} & &
\end{tikzcd}$$

where the bottom identity 2-cell above is established in Lemma 7.11 in Appendix. Similarly one has the 2-cell in SPC

3.29 $u^2{}_f$:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_x"] \arrow[dr, "\tilde{f}"'] \arrow[r, phantom, shift right=4ex, "\Uparrow"] & \mathcal{A}_{x,x} \arrow[d, "\mathcal{A}(1,f)"] \\
 & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]}
\end{tikzcd}$$

that corresponds the 2-cell

$$(\lambda_{x,y})_f : f \to f \circ 1_x$$

and is the pasting

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_x"] \arrow[rr, bend right=50, "v"'] \arrow[rrrr, bend right=70, "\tilde{f}"'] & \mathcal{A}_{x,x} \arrow[r, "\mathcal{A}(-,y)"'] \arrow[rrr, bend left=40, "\mathcal{A}(1,f)"] \arrow[d, phantom, "\Uparrow r'"] & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,y}]} \arrow[r, "{[\tilde{f},1]}"] \arrow[u, phantom, "="] \arrow[d, phantom, "="] & {[\mathcal{I}, \mathcal{A}_{x,y}]} \arrow[r, "ev_\star"] & \mathcal{A}_{x,y} \\
 & {} & {} & &
\end{tikzcd}$$

Given two SPC-functors $F, G : \mathcal{A} \to \mathcal{B}$, a *$SPC$-natural transformation* $(\sigma, \kappa) : F \to G : \mathcal{A} \to \mathcal{B}$ consists of a collection of *strict* arrows $\sigma_x : \mathcal{I} \to \mathcal{B}(Fx, Gx)$ (or 1-cells $\sigma_x : Fx \to Gx$ in $\mathcal{B}_0$),

indexed by objects x of $\mathcal{A}$ together with a collection of 2-cells $\kappa_{x,y}$ in SPC for objects x,y of $\mathcal{A}$ as follows

3.30

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "F_{x,y}"] \arrow[d, "G_{x,y}"'] & \mathcal{B}_{Fx,Fy} \arrow[d, "\mathcal{B}(1,\sigma_y)"] \\
\mathcal{B}_{Gx,Gy} \arrow[r, "\mathcal{B}(\sigma_x,1)"'] \arrow[ur, Rightarrow] & \mathcal{B}_{Fx,Gy}
\end{tikzcd}$$

and that satisfies the two coherence conditions 3.31, 3.32 and below.

3.31 *For any object x, y and z in $\mathcal{A}$, the 2-cells Ξ_1, Ξ_2, Ξ_3, Ξ_4, Ξ_5, Ξ_6, Ξ_7 and Ξ_8 below satisfy the equality*

$$\Xi_2 \circ \Xi_1 = \Xi_8 \circ (\Xi_7)^{-1} \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ (\Xi_3)^{-1}.$$

Ξ_1 *is*

$$\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, "G_{y,z}"] \arrow[d, "\mathcal{A}(x,-)"'] & \mathcal{B}_{Gy,Gz} \arrow[r, "\mathcal{B}(Gx,-)"] \arrow[d, phantom, "\Downarrow {G'}^2_{x,y,z}"] & {[\mathcal{B}_{Gx,Gy}, \mathcal{B}_{Gx,Gz}]} \arrow[d, "{[G_{x,y},1]}"] & \\
{[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]} \arrow[rr, "{[1,G_{x,z}]}"'] & & {[\mathcal{A}_{x,y}, \mathcal{B}_{Gx,Gz}]} \arrow[r, "{[1,\mathcal{B}(\sigma_x,1)]}"'] & {[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Gz}]}
\end{tikzcd}$$

Ξ_2 *is*

$$\begin{tikzcd}
& & {[\mathcal{A}_{x,y}, \mathcal{B}_{Gx,Gz}]} \arrow[dd, phantom, "\Downarrow {[1,\kappa_{x,z}]}"] \arrow[dr, "{[1,\mathcal{B}(\sigma_x,1)]}"] & \\
\mathcal{A}_{y,z} \arrow[r, "\mathcal{A}(x,-)"] & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]} \arrow[ur, "{[1,G_{x,z}]}"] \arrow[dr, "{[1,F_{x,z}]}"'] & & {[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Gz}]} \\
& & {[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Fz}]} \arrow[ur, "{[1,\mathcal{B}(1,\sigma_z)]}"'] &
\end{tikzcd}$$

Ξ_3 *is*

$$\begin{tikzcd}
& & {[\mathcal{B}_{Gx,Gy}, \mathcal{B}_{Gx,Gz}]} \arrow[dr, "{[1,\mathcal{B}(\sigma_x,1)]}"] & & \\
\mathcal{A}_{y,z} \arrow[r, "G_{y,z}"] & \mathcal{B}_{Gy,Gz} \arrow[ur, "\mathcal{B}(Gx,-)"] \arrow[dr, "\mathcal{B}(Fx,-)"'] & & {[\mathcal{B}_{Gx,Gy}, \mathcal{B}_{Fx,Gz}]} \arrow[r, "{[G_{x,y},1]}"] & {[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Gz}]} \\
& & {[\mathcal{B}_{Fx,Gx}, \mathcal{B}_{Fx,Gz}]} \arrow[uu, phantom, "\Uparrow c^1_{\mathcal{B}(\sigma_x,1),y,z}"] \arrow[ur, "{[\mathcal{B}(\sigma_x,1),1]}"'] & &
\end{tikzcd}$$

Ξ_4 *is*

$$\begin{tikzcd}
& & & {[\mathcal{B}_{Gx,Gy}, \mathcal{B}_{Fx,Gz}]} \arrow[dd, phantom, "\Downarrow {[\kappa_{x,y},1]}"] \arrow[dr, "{[G_{x,y},1]}"] & \\
\mathcal{A}_{y,z} \arrow[r, "G_{y,z}"] & \mathcal{B}_{Gy,Gz} \arrow[r, "\mathcal{B}(Fx,-)"] & {[\mathcal{B}_{Fx,Gy}, \mathcal{B}_{Fx,Gz}]} \arrow[ur, "{[\mathcal{B}(\sigma_x,1),1]}"] \arrow[dr, "{[\mathcal{B}(1,\sigma_y),1]}"'] & & {[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Gz}]} \\
& & & {[\mathcal{B}_{Fx,Fy}, \mathcal{B}_{Fx,Gz}]} \arrow[ur, "{[F_{x,y},1]}"'] &
\end{tikzcd}$$

Ξ_5 *is*

$$
\begin{tikzcd}
 & & {[\mathcal{B}_{Fx,Gy},\mathcal{B}_{Fx,Gz}]} \arrow[dd, phantom, "\Downarrow c^2{}_{Fx,\sigma_y,Gz}"] \arrow[dr, "{[\mathcal{B}(1,\sigma_y),1]}"] & & \\
\mathcal{A}_{y,z} \arrow[r, "G_{y,z}"] & \mathcal{B}_{Gy,Gz} \arrow[ur, "{\mathcal{B}(Fx,-)}"] \arrow[dr, "{\mathcal{B}(\sigma_y,1)}"'] & & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Gz}]} \arrow[r, "{[F_{x,y},1]}"] & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Gz}]} \\
 & & \mathcal{B}_{Fy,Gz} \arrow[ur, "{\mathcal{B}(Fx,-)}"'] & &
\end{tikzcd}
$$

Ξ_6 *is*

$$
\begin{tikzcd}
 & \mathcal{B}_{Gy,Gz} \arrow[dd, phantom, "\Downarrow \kappa_{y,z}"] \arrow[dr, "{\mathcal{B}(\sigma_y,1)}"] & & & \\
\mathcal{A}_{y,z} \arrow[ur, "G_{y,z}"] \arrow[dr, "F_{y,z}"'] & & \mathcal{B}_{Fy,Gz} \arrow[r, "{\mathcal{B}(Fx,-)}"] & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Gz}]} \arrow[r, "{[F_{x,y},1]}"] & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Gz}]} \\
 & \mathcal{B}_{Fy,Fz} \arrow[ur, "{\mathcal{B}(1,\sigma_z)}"'] & & &
\end{tikzcd}
$$

Ξ_7 *is*

$$
\begin{tikzcd}
 & & \mathcal{B}_{Fy,Gz} \arrow[dr, "{\mathcal{B}(Fx,-)}"] & & \\
\mathcal{A}_{y,z} \arrow[r, "F_{y,z}"] & \mathcal{B}_{Fy,Fz} \arrow[ur, "{\mathcal{B}(1,\sigma_z)}"] \arrow[dr, "{\mathcal{B}(Fx,-)}"'] & & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Gz}]} \arrow[r, "{[F_{x,y},1]}"] & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Gz}]} \\
 & & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}]} \arrow[uu, phantom, "\Uparrow c^3{}_{Fx,Fy,\sigma_z}"] \arrow[ur, "{[1,\mathcal{B}(1,\sigma_z)]}"'] & &
\end{tikzcd}
$$

Ξ_8 *is*

$$
\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, "F_{y,z}"] \arrow[d, "{\mathcal{A}(x,-)}"'] & \mathcal{B}_{Fy,Fz} \arrow[r, "{\mathcal{B}(Fx,-)}"] \arrow[d, phantom, "\Downarrow F'^2_{x,y,z}"] & {[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}]} \arrow[d, "{[F_{x,y},1]}"] & \\
{[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]} \arrow[rr, "{[1,F]}"'] & & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Fz}]} \arrow[r, "{[1,\mathcal{B}(1,\sigma_z)]}"] & {[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Gz}]}
\end{tikzcd}
$$

3.32 *For any object x of $\mathcal{A}$, the 2-cells*

$$
\begin{tikzcd}
\mathcal{I} \arrow[r, "u"] \arrow[rr, bend left=60, "u"] \arrow[rrr, bend left=80, "\sigma_x"] & \mathcal{A}_{x,x} \arrow[u, phantom, "\Downarrow F^0"] \arrow[r, "F_{x,x}"'] & \mathcal{B}_{Fx,Fx} \arrow[u, phantom, "\Downarrow u^2{}_{\sigma_x}"] \arrow[r, "{\mathcal{B}(1,\sigma_x)}"'] & \mathcal{B}_{Fx,Gx}
\end{tikzcd}
$$

and

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_{Gx}"] \arrow[d, dash, "id"'] \arrow[rr, bend left=50, "\sigma_x"] \arrow[d, phantom, shift left=6, "\Downarrow G^0"] & \mathcal{B}_{Gx,Gx} \arrow[r, "\mathcal{B}(\sigma_x{,}1)"] \arrow[u, phantom, "\Downarrow u^1{}_{\sigma_x}"] \arrow[dr, phantom, "\Rightarrow \kappa"] & \mathcal{B}_{Fx,Gx} \\
\mathcal{I} \arrow[r, "u_x"'] & \mathcal{A}_{x,x} \arrow[r, "F_{x,x}"'] \arrow[u, "G_{x,x}"] & \mathcal{B}_{Fx,Fx} \arrow[u, "\mathcal{B}(1{,}\sigma_x)"']
\end{tikzcd}$$

are equal.

4 SPC-categories via the tensor

In this section we give definitions of SPC-categories and SPC-functors that rely on the tensor product in SPC.

A *SPC-category* $(\mathcal{A}, u, c, \alpha, \rho, \lambda)$ consists of the following data:
- As before: a small set of objects with a map sending any pair x,y of objects to an object $\mathcal{A}_{x,y}$ of SPC;
- Collections of *strict* morphisms $u_x : \mathcal{I} \to \mathcal{A}_{x,x}$ and $c_{x,y,z} : \mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \to \mathcal{A}_{x,z}$ indexed by objects of $\mathcal{A}$ with collections of 2-cells $\alpha_{x,y,z,t}$, $\rho_{x,y}$ and $\lambda_{x,y}$ in SPC indexed by objects of $\mathcal{A}$ and as follows

4.1

$$\begin{tikzcd}
(\mathcal{A}_{z,t} \otimes \mathcal{A}_{y,z}) \otimes \mathcal{A}_{x,y} \arrow[rr, "A'"] \arrow[d, "c_{y,z,t} \otimes 1"'] & & \mathcal{A}_{z,t} \otimes (\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y}) \arrow[d, "1 \otimes c_{x,y,z}"] \arrow[dll, Rightarrow, "\alpha_{x,y,z,t}"'] \\
\mathcal{A}_{y,t} \otimes \mathcal{A}_{x,y} \arrow[dr, "c_{x,y,t}"'] & & \mathcal{A}_{z,t} \otimes \mathcal{A}_{x,z} \arrow[dl, "c_{x,z,t}"] \\
& \mathcal{A}_{x,t} &
\end{tikzcd}$$

4.2

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "R'"] \arrow[d, dash, "id"'] \arrow[d, phantom, shift left=8, "\xRightarrow{\rho_{x,y}}"] & \mathcal{A}_{x,y} \otimes \mathcal{I} \arrow[d, "1 \otimes u_x"] \\
\mathcal{A}_{x,y} & \mathcal{A}_{x,y} \otimes \mathcal{A}_{x,x} \arrow[l, "c_{x,x,y}"]
\end{tikzcd}$$

and

4.3

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "L'"] \arrow[d, dash, "id"'] \arrow[d, phantom, shift left=8, "\xRightarrow{\lambda_{x,y}}"] & \mathcal{I} \otimes \mathcal{A}_{x,y} \arrow[d, "u_y \otimes 1"] \\
\mathcal{A}_{x,y} & \mathcal{A}_{y,y} \otimes \mathcal{A}_{x,y} \arrow[l, "c_{x,y,y}"]
\end{tikzcd}$$

Those satisfy the coherence Axioms 4.4 and 4.5 below.

4.4 *For any objects x,y,z,t and u of $\mathcal{A}$, the 2-cells*

$$
\begin{tikzcd}
((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \arrow[r, "A'"] \arrow[d, "(c_{z,t,u}\otimes 1)\otimes 1"'] \arrow[dr, phantom, "="] & (\mathcal{A}_{t,u}\mathcal{A}_{z,t})(\mathcal{A}_{y,z}\mathcal{A}_{x,y}) \arrow[r, dash, "id"] \arrow[d, "c_{z,t,u}\otimes 1"] & (\mathcal{A}_{t,u}\mathcal{A}_{z,t})(\mathcal{A}_{y,z}\mathcal{A}_{x,y}) \arrow[r, "A'"] \arrow[d, "1\otimes c_{x,y,z}"] \arrow[dr, phantom, "="] & \mathcal{A}_{t,u}(\mathcal{A}_{z,t}(\mathcal{A}_{y,z}\mathcal{A}_{x,y})) \arrow[d, "1\otimes(1\otimes c_{x,y,z})"] \\
(\mathcal{A}_{z,u}\mathcal{A}_{y,z})\mathcal{A}_{x,y} \arrow[r, "A'"] \arrow[d, "c_{y,z,t}\otimes 1"'] & \mathcal{A}_{z,u}(\mathcal{A}_{y,z}\mathcal{A}_{x,y}) \arrow[r, phantom, "="] \arrow[d, "1\otimes c_{x,y,z}"] \arrow[ddl, Rightarrow, "\alpha_{x,y,z,u}"] & (\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{x,z} \arrow[r, "A'"] \arrow[d, "c_{z,t,u}\otimes 1"] & \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{x,y}) \arrow[d, "1\otimes c_{x,y,t}"] \arrow[ddl, Rightarrow, "\alpha_{x,z,t,u}"] \\
\mathcal{A}_{y,u}\mathcal{A}_{x,y} \arrow[d, "c_{x,y,u}"'] & \mathcal{A}_{z,u}\mathcal{A}_{x,z} \arrow[r, dash, "id"] \arrow[d, "c_{x,z,u}"'] \arrow[dr, phantom, "="] & \mathcal{A}_{z,u}\mathcal{A}_{x,z} \arrow[d, "c_{x,z,u}"'] & \mathcal{A}_{t,u}\mathcal{A}_{x,t} \arrow[d, "c_{x,t,u}"] \\
\mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \arrow[r, "A'\otimes 1"] \arrow[d, "(c_{z,t,u}\otimes 1)\otimes 1"'] & ((\mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z}))\mathcal{A}_{x,y} \arrow[r, "A'"] \arrow[d, "(1\otimes c_{y,z,t})\otimes 1"] \arrow[dr, phantom, "="] \arrow[ddl, Rightarrow, "\alpha_{y,z,t,u}\otimes 1"] & \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}) \arrow[r, "1\otimes A'"] \arrow[d, "1\otimes(c_{y,z,t}\otimes 1)"] & \mathcal{A}_{t,u}(\mathcal{A}_{z,t}(\mathcal{A}_{y,z}\mathcal{A}_{x,y})) \arrow[d, "1\otimes(1\otimes c_{x,y,z})"] \arrow[ddl, Rightarrow, "1\otimes\alpha_{x,y,z,t}"] \\
(\mathcal{A}_{z,u}\mathcal{A}_{y,z})\mathcal{A}_{x,y} \arrow[d, "c_{y,z,u}\otimes 1"'] & (\mathcal{A}_{t,u}\mathcal{A}_{y,t})\mathcal{A}_{x,y} \arrow[r, "A'"] \arrow[d, "c_{y,t,u}\otimes 1"] & \mathcal{A}_{t,u}(\mathcal{A}_{y,t}\mathcal{A}_{x,y}) \arrow[d, "1\otimes c_{x,y,t}"] \arrow[ddl, Rightarrow, "\alpha_{x,y,t,u}"] & \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{x,z}) \arrow[d, "1\otimes c_{x,z,t}"] \\
\mathcal{A}_{y,u}\mathcal{A}_{x,y} \arrow[r, dash, "id"] \arrow[d, "c_{x,y,u}"'] \arrow[dr, phantom, "="] & \mathcal{A}_{y,u}\mathcal{A}_{x,y} \arrow[d, "c_{x,y,u}"'] & \mathcal{A}_{t,u}\mathcal{A}_{x,t} \arrow[r, dash, "id"] \arrow[d, "c_{x,t,u}"'] \arrow[dr, phantom, "="] & \mathcal{A}_{t,u}\mathcal{A}_{x,t} \arrow[d, "c_{x,t,u}"] \\
\mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u} \arrow[r, dash, "id"'] & \mathcal{A}_{x,u}
\end{tikzcd}
$$

are equal. Note that the domains of the above 2-cells are equal since $(1\otimes A')\circ A'\circ(A'\otimes 1) = A'\circ A'$ by Lemma [Sch08]-19.10.

4.5 *For any objects x, y, z of $\mathcal{A}$, the 2-cells*

$\Xi_1 =$

$$
\begin{tikzcd}
\mathcal{A}_{y,z}\otimes\mathcal{A}_{x,y} \arrow[rr, dash, "id"] \arrow[d, "R'\otimes 1"'] & \arrow[d, phantom, "\Downarrow \rho_{y,z}\otimes 1"] & \mathcal{A}_{y,z}\otimes\mathcal{A}_{x,y} \arrow[r, "c_{x,y,z}"] & \mathcal{A}_{x,z} \\
(\mathcal{A}_{y,z}\otimes\mathcal{I})\otimes\mathcal{A}_{x,y} \arrow[rr, "(1\otimes u_y)\otimes 1"'] & {} & (\mathcal{A}_{y,z}\otimes\mathcal{A}_{y,y})\otimes\mathcal{A}_{x,y} \arrow[u, "c_{y,y,z}\otimes 1"'] &
\end{tikzcd}
$$

$\Xi_2 =$

$$
\begin{tikzcd}
\mathcal{A}_{y,z}\otimes\mathcal{A}_{x,y} \arrow[rr, dash, "id"] \arrow[d, "1\otimes L'"'] & \arrow[d, phantom, "\Downarrow 1\otimes\lambda_{x,y}"] & \mathcal{A}_{y,z}\otimes\mathcal{A}_{x,y} \arrow[r, "c_{x,y,z}"] & \mathcal{A}_{x,z} \\
\mathcal{A}_{y,z}\otimes(\mathcal{I}\otimes\mathcal{A}_{x,y}) \arrow[rr, "1\otimes(u_y\otimes 1)"'] & {} & \mathcal{A}_{y,z}\otimes(\mathcal{A}_{y,y}\otimes\mathcal{A}_{x,y}) \arrow[u, "1\otimes c_{x,y,y}"'] &
\end{tikzcd}
$$

and

$\Xi_3 =$

$$\begin{tikzcd}
 & (\mathcal{A}_{y,z} \otimes \mathcal{I}) \otimes \mathcal{A}_{x,y} \arrow[r, "(1\otimes u_y)\otimes 1"] \arrow[dd, "A'"] & (\mathcal{A}_{y,z} \otimes \mathcal{A}_{y,y}) \otimes \mathcal{A}_{x,y} \arrow[r, "c_{y,y,z}\otimes 1"] \arrow[dd, "A'"] & \mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \arrow[dr, "c_{x,y,z}"] & \\
\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \arrow[ur, "R'\otimes 1"] \arrow[dr, "1\otimes L'"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} & & \mathcal{A}_{x,z} \\
 & \mathcal{A}_{y,z} \otimes (\mathcal{I} \otimes \mathcal{A}_{x,y}) \arrow[r, "1\otimes(u_y\otimes 1)"'] & \mathcal{A}_{y,z} \otimes (\mathcal{A}_{y,y} \otimes \mathcal{A}_{x,y}) \arrow[r, "1\otimes c_{x,y,y}"'] \arrow[uur, Rightarrow, "\alpha_{x,y,y,z}"] & \mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \arrow[ur, "c_{x,y,z}"'] &
\end{tikzcd}$$

satisfy the equality $\Xi_1 = \Xi_3 * \Xi_2$

Let us justify the equivalence of the definitions of SPC-categories. We define the following bijective correspondence between data involved the definitions. Arrows $\mathcal{A}(x,-)_{y,z} : \mathcal{A}_{y,z} \to [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]$ and $c_{x,y,z} : \mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \to \mathcal{A}_{x,z}$ correspond via the adjunction 2.5. By Lemma [Sch08]-19.6 one has a bijective correspondence between 2-cells of the kind $\alpha_{x,y,z,t}$ and 2-cells $\alpha'_{x,y,z,t}$, the later being images by $Rn \circ Rn$ of the first ones. The codomains of the 2-cells $\rho_{x,y}$ and $\rho'_{x,y}$ are equal by Lemma 7.9, and these 2-cells correspond when are equal. The codomains of the 2-cells $\lambda_{x,y}$ and $\lambda'_{x,y}$ are equal by Lemma 7.8 and these 2-cells correspond when they are equal. For such corresponding data, the proofs of the equivalence of Axioms 4.4 and 3.14 rely on the adjunction 2.5. The 2-cells of Axiom 4.4 have images by $Rn \circ Rn \circ Rn$ the two 2-cells of Axioms 3.14 and their common domain is a strict arrow with strict images by Rn and $Rn \circ Rn$. Computation details are in Appendix in 7.27. The proof of the equivalence of Axioms 4.5 and Axioms 3.15 is similar. The 2-cells Ξ_1 of Axiom 4.5 has a strict domain and its image by Rn is ρ' whereas the 2-cell $\Xi_3 \circ \Xi_2$ has image by Rn the second 2-cell of Axiom 3.15. Computation details are in Appendix in 7.28.

We have an alternative definition for the SPC-functors with the tensor in SPC.

Given two arbitrary SPC-categories $\mathcal{A}$ and $\mathcal{B}$, a *SPC-functor* $F : \mathcal{A} \to \mathcal{B}$ consists of the following data:
- A map F sending objects of $\mathcal{A}$ to objects of $\mathcal{B}$;
- For any objects x,y of $\mathcal{A}$, and arrow $F_{x,y} : \mathcal{A}(x,y) \to \mathcal{B}(Fx, Fy)$ in SPC;
- Collections of 2-cells of SPC: the $F^2_{x,y}$, indexed by pair of objects x,y of $\mathcal{A}$ and the F^0_x, indexed by objects x of $\mathcal{A}$, as follows

4.6

$$\begin{tikzcd}
\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \arrow[r, "F_{y,z}\otimes F_{x,y}"] \arrow[d, "c"'] & \mathcal{B}_{Fy,Fz} \otimes \mathcal{B}_{Fx,Fy} \arrow[d, "c"] \arrow[dl, Rightarrow, "F^2_{x,y,z}"] \\
\mathcal{A}_{x,z} \arrow[r, "F_{x,z}"'] & \mathcal{B}_{Fx,Fz}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u_x"] \arrow[dr, "u_{Fx}"'] \arrow[r, phantom, shift right=3ex, "\Uparrow F^0_x"] & \mathcal{A}_{x,x} \arrow[d, "F_{x,x}"] \\
 & \mathcal{B}_{Fx,Fx}
\end{tikzcd}$$

and that satisfy the coherence conditions 4.7, 4.8 and 4.9 below.

4.7 *For any objects x,y,z,t of $\mathcal{A}$, the 2-cells*

$$
\begin{tikzcd}
(\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z})\otimes\mathcal{A}_{x,y} \arrow[r, "c\otimes 1"] \arrow[d, "(F_{z,t}\otimes F_{y,z})\otimes F_{x,y}"'] & \mathcal{A}_{y,t}\otimes\mathcal{A}_{x,y} \arrow[r, "c"] \arrow[d, "F_{y,t}\otimes F_{x,y}"] & \mathcal{A}_{x,t} \arrow[d, "F_{x,t}"] \\
(\mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fy,Fz})\otimes\mathcal{B}_{Fx,Fy} \arrow[r, "c\otimes 1"'] \arrow[d, "A'"'] \arrow[ur, phantom, "\Nearrow F^2_{y,z,t}\otimes 1"] & \mathcal{B}_{Fy,Ft}\otimes\mathcal{B}_{Fx,Fy} \arrow[r, "c"] \arrow[ur, phantom, "\Nearrow F^2_{x,y,t}"] & \mathcal{B}_{Fx,Ft} \arrow[d, no head, "id"] \\
\mathcal{B}_{Fz,Ft}\otimes(\mathcal{B}_{Fy,Fz}\otimes\mathcal{B}_{Fx,Fy}) \arrow[r, "1\otimes c"'] \arrow[urr, phantom, "\Nearrow \alpha_{Fx,Fy,Fz,Ft}"] & \mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fx,Fz} \arrow[r, "c"'] & \mathcal{B}_{Fx,Ft}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z})\otimes\mathcal{A}_{x,y} \arrow[r, "c\otimes 1"] \arrow[d, "A'"'] & \mathcal{A}_{y,t}\otimes\mathcal{A}_{x,y} \arrow[r, "c"] & \mathcal{A} \arrow[d, no head, "id"] \\
\mathcal{A}_{z,t}\otimes(\mathcal{A}_{y,z}\otimes\mathcal{A}_{x,y}) \arrow[r, "1\otimes c"'] \arrow[d, "F_{z,t}\otimes(F_{y,z}\otimes F_{x,y})"'] \arrow[urr, phantom, "\Nearrow \alpha_{x,y,z,t}"] & \mathcal{A}_{z,t}\otimes\mathcal{A}_{x,z} \arrow[r, "c"] \arrow[d, "F_{z,t}\otimes F_{x,z}"] & \mathcal{A}_{x,t} \arrow[d, "F_{x,t}"] \\
\mathcal{B}_{Fz,Ft}\otimes(\mathcal{B}_{Fy,Fz}\otimes\mathcal{B}_{Fx,Fy}) \arrow[r, "1\otimes c"'] \arrow[ur, phantom, "\Nearrow 1\otimes F^2_{x,y,z}"] & \mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fx,Fz} \arrow[r, "c"'] \arrow[ur, phantom, "\Nearrow F^2_{x,z,t}"] & \mathcal{B}_{Fx,Ft}
\end{tikzcd}
$$

are equal.

4.8 *For any objects x,y of $\mathcal{A}$, the 2-cells*

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[r, "F_{x,y}"] \arrow[d, "R'"'] \arrow[dr, phantom, "="] & \mathcal{B}_{Fx,Fy} \arrow[d, "R'"] \arrow[ddd, bend left=60, no head, "id"] \\
\mathcal{A}_{x,y}\otimes\mathcal{I} \arrow[r, "F_{x,y}\otimes 1"] \arrow[d, "1\otimes u"'] & \mathcal{B}_{Fx,Fy}\otimes\mathcal{I} \arrow[d, "1\otimes u"] \arrow[dl, phantom, "\Swarrow 1\otimes F^0_x"] \arrow[d, phantom, shift left=8ex, "\Leftarrow \rho_{Fx,Fy}"] \\
\mathcal{A}_{x,y}\otimes\mathcal{A}_{x,x} \arrow[r, "F_{x,y}\otimes F_{x,x}"] \arrow[d, "c"'] & \mathcal{B}_{Fx,Fy}\otimes\mathcal{B}_{Fx,Fx} \arrow[d, "c"] \arrow[dl, phantom, "\Swarrow F^2_{x,x,y}"] \\
\mathcal{A}_{x,y} \arrow[r, "F_{x,y}"'] & \mathcal{B}_{Fx,Fy}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[rr, no head, "id"] \arrow[d, "R'"'] \arrow[drr, phantom, "\Downarrow \rho_{x,y}"] & & \mathcal{A}_{x,y} \arrow[r, "F"] & \mathcal{B}_{Fx,Fy} \\
\mathcal{A}_{x,y}\otimes\mathcal{I} \arrow[rr, "1\otimes u_x"'] & & \mathcal{A}_{x,y}\otimes\mathcal{A}_{x,x} \arrow[u, "c"']
\end{tikzcd}
$$

are equal.

4.9 *For any objects x,y of $\mathcal{A}$, the 2-cells*

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[rr, "F"] \arrow[d, "L'"'] \arrow[drr, phantom, "="] & & \mathcal{B}_{Fx,Fy} \arrow[d, "L'"] \arrow[ddd, bend left=60, "id"] \arrow[ddd, phantom, bend left=30, "\Leftarrow \lambda_{Fx,Fy}"] \\
\mathcal{I} \otimes \mathcal{A}_{x,y} \arrow[rr, "1 \otimes F_{x,y}"] \arrow[d, "u_y \otimes 1"'] \arrow[drr, phantom, "\Swarrow F^0_{x,y} \otimes 1"] & & \mathcal{I} \otimes \mathcal{B}_{Fx,Fy} \arrow[d, "u_{Fy} \otimes 1"] \\
\mathcal{A}_{y,y} \otimes \mathcal{A}_{x,y} \arrow[rr, "F_{y,y} \otimes F_{x,y}"] \arrow[d, "c"'] \arrow[drr, phantom, "\Swarrow F^2_{x,y,y}"] & & \mathcal{B}_{Fy,Fy} \otimes \mathcal{B}_{Fx,Fy} \arrow[d, "c"] \\
\mathcal{A} \arrow[rr, "F_{x,y}"'] & & \mathcal{B}_{Fx,Fy}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}_{x,y} \arrow[rr, "id"] \arrow[d, "L'"'] \arrow[drr, phantom, "\Downarrow \lambda"] & & \mathcal{A}_{x,y} \arrow[r, "F_{x,y}"] & \mathcal{B}_{Fx,Fy} \\
\mathcal{I} \otimes \mathcal{A}_{x,y} \arrow[rr, "u_y \otimes 1"'] & & \mathcal{A}_{y,y} \otimes \mathcal{A}_{x,y} \arrow[u, "c"'] &
\end{tikzcd}$$

are equal.

The adjunction 2.5 gives a bijective correspondence between 2-cells F^2 as in 4.6 and 2-cells ${F'}^2$ as in 3.16. For such coresponding data, it turns out that Axioms 3.18 and 4.7 are equivalent, this is proved in 7.29 in Appendix, Axioms 3.19 and 4.8 are equivalent, this is proved in 7.30 in Appendix, and Axioms 3.20 and 4.9 are equivalent, this is proved in 7.31 in Appendix.

We say that a SPC-category $\mathcal{A}$ as above, is *strict* if and only if the arrows $\mathcal{A}(x,-)$ are strict in SPC and the 2-cells α, ρ, λ, or equivalently the 2-cells α', ρ' and λ', are all identities. We have also the notion of *strict* SPC-functors $(F, F^0, F^2) : \mathcal{A} \to \mathcal{B}$: they are the ones for which the components $F_{x,y} : \mathcal{A}_{x,y} \to \mathcal{B}_{Fx,Fy}$ are strict arrows in SPC and for which the 2-cells of the collections F^0 and ${F'}^2$ are identities. Note that for an F as above with $\mathcal{A}$ strict the ${F'}^2$ are identities if and only if the F^2 are.

5 First examples

One-point enrichments are of particular interest and named *2-rings.* As mentioned by M.Dupont in his thesis [Dup08], they are also the *categorical rings* defined by Jibladze and Pirashvili [JiPi07], those are also known to be the *Ann-categories* of [Qu87]. Given a 2-ring $\mathcal{A}$, we shall write simply $\mathcal{A}$ for the hom $\mathcal{A}(*,*)$ of its unique object $*$. Therefore the formal definition of a 2-ring $(\mathcal{A}, c, u, \alpha, \rho, \lambda)$, as a "weak" monoid in SPC, namely a Picard category $\mathcal{A}$ with a *multiplication* $c : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a *unit* $u : \mathcal{I} \to \mathcal{A}$ (which are strict arrows!) and appropriate 2-cells α, ρ and λ, is obtained by removing the subscripts $x, y, z, ...$ from the definitions of SPC-categories. We shall use the alternative definition of a 2-ring $(\mathcal{A}, c', u, \alpha', \rho', \lambda')$ not using the tensor obtained similarly by forgetting subscripts and where multiplication $c' : \mathcal{A} \to [\mathcal{A}, \mathcal{A}]$ denotes the unique arrow $\mathcal{A}_{*,-}$.

The following definition of 2-rings can be obtained by written explicitly all linearity conditions from the definition of enriched categories and functors. It is equivalent and very close to that of Jibladze and Pirashvili [JiPi07] (for their categorical rings). Detailed explanations that the 2-rings with their morphisms in the sense below are just one-point SPC-categories with their functors is given in Appendix-7.33 and 7.34.

Definition 5.1 *A* 2-ring *consists of a symmetric Picard category $(\mathcal{A}, j)$, where $\mathcal{A}$ is denoted additively $(\mathcal{A}, +, 0, ass, r, l, s)$, together with a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, denoted by a multiplication ".", an object 1 of $\mathcal{A}$ and natural isomorphisms*

$$\tilde{\alpha}_{a,b,c} : (a.b).c \to a.(b.c),$$

$$\tilde{\rho}_a : a.1 \to a,$$
$$\tilde{\lambda}_a : 1.a \to a,$$
$$\underline{a}_{b,b'} : a.b + a.b' \to a.(b + b'),$$
$$\overline{b}_{a,a'} : a.b + a'.b \to (a + a').b,$$

such that the data $(\mathcal{A}, ., 1, \tilde{\alpha}, \tilde{\rho}, \tilde{\lambda})$ defines a monoidal structure on $\mathcal{A}$ and the diagrams below from 5.2 to 5.11 commute for all possible objects of $\mathcal{A}$.

5.2

$$\begin{tikzcd}
a.b + (a.b' + a.b'') \arrow[r, "ass"] \arrow[d, "id+\underline{a}_{b',b''}"'] & (a.b + a.b') + a.b'' \arrow[d, "\underline{a}_{b,b'}+id"] \\
a.b + a.(b' + b'') \arrow[d, "\underline{a}_{b,b'+b''}"'] & a.(b + b') + a.d \arrow[d, "\underline{a}_{b+b',b''}"] \\
a.(b.(b' + b'')) \arrow[r, "a.ass"'] & a.((b + b') + b'')
\end{tikzcd}$$

5.3

$$\begin{tikzcd}
a.b + (a'.b + a''.b) \arrow[r, "ass"] \arrow[d, "id+\overline{b}_{a',a''}"'] & (a.b + a'.b) + a''.b \arrow[d, "\overline{b}_{a,a'}+id"] \\
a.b + (a' + a'').b \arrow[d, "\overline{b}_{a,a'+a''}"'] & (a + a').b + a''.b \arrow[d, "\overline{b}_{a+a',a''}"] \\
(a + (a' + a'')).b \arrow[r, "ass.b"'] & ((a + a') + a'').b
\end{tikzcd}$$

5.4

$$\begin{tikzcd}
a.b + a.b' \arrow[r, "\underline{a}_{b,b'}"] \arrow[d, "s"'] & a.(b + b') \arrow[d, "a.s"] \\
a.b' + a.b \arrow[r, "\underline{a}_{b',b}"'] & a.(b' + b).
\end{tikzcd}$$

5.5

$$\begin{tikzcd}
a.b + a'.b \arrow[r, "\overline{b}_{a,a'}"] \arrow[d, "s"'] & (a + a').b \arrow[d, "s.b"] \\
a'.b + a.b \arrow[r, "\overline{b}_{a',a}"'] & (a' + a).b
\end{tikzcd}$$

5.6

$$
\begin{tikzcd}[column sep=small]
(a.b + a.b') + (a'.b + a'.b') \arrow[rr, dash, "\cong"] \arrow[d, "\underline{a}_{b,b'} + \underline{a'}_{b,b'}"'] & & (a.b + a'.b) + (a'.b + a'.b') \arrow[d, "\overline{b}_{a,a'} + \overline{b'}_{a,a'}"] \\
a.(b + b') + a'.(b + b') \arrow[dr, "\overline{b+b'}_{a,a'}"'] & & (a + a').b + (a + a').b' \arrow[dl, "\underline{a+a'}_{b,b'}"] \\
& (a + a').(b + b') &
\end{tikzcd}
$$

5.7

$$
\begin{tikzcd}
(a.b).c + (a'.b).c \arrow[r, "\overline{c}_{a.b,a'.b}"] \arrow[d, "\tilde{\alpha}_{a,b,c} + \tilde{\alpha}_{a',b,c}"'] & ((a.b) + (a'.b)).c \arrow[r, "\overline{b}_{a,a'}.c"] & ((a + a').b).c \arrow[d, "\tilde{\alpha}_{a+a',b,c}"] \\
a.(b.c) + a'.(b.c) \arrow[rr, "\overline{b.c}_{a,a'}"'] & & (a + a').(b.c)
\end{tikzcd}
$$

5.8

$$
\begin{tikzcd}
(a.b).c + (a.b').c \arrow[r, "\overline{c}_{a.b,a.b'}"] \arrow[d, "\tilde{\alpha}_{a,b,c} + \tilde{\alpha}_{a,b',c}"'] & ((a.b) + (a.b')).c \arrow[r, "\underline{a}_{b,b'}.c"] & (a.(b + b')).c \arrow[d, "\tilde{\alpha}_{a,b+b',c}"] \\
a.(b.c) + a.(b.c') \arrow[r, "\underline{a}_{b.c,b.c'}"'] & a.(b.c + b.c') \arrow[r, "a.\overline{c}_{b,b'}"'] & a.((b + b').c)
\end{tikzcd}
$$

5.9

$$
\begin{tikzcd}
(a.b).c + (a.b).c' \arrow[rr, "\underline{a.b}_{c,c'}"] \arrow[d, "\tilde{\alpha}_{a,b,c} + \tilde{\alpha}_{a,b,c'}" description] & & (a.b).(c + c') \arrow[d, "\tilde{\alpha}_{a,b,c+c'}" description] \\
a.(b.c) + a.(b.c') \arrow[r, "\underline{a}_{b.c,b.c'}"'] & a.(b.c + b.c') \arrow[r, "a.\underline{b}_{c,c'}"'] & a.(b.(c + c'))
\end{tikzcd}
$$

5.10

$$
\begin{tikzcd}
(a + b).1 \arrow[r, "\tilde{\rho}_{a+b}"] \arrow[d, "\overline{1}_{a,b}"'] & a + b \arrow[dl, equal] \\
a + b &
\end{tikzcd}
$$

5.11

$$
\begin{tikzcd}
1.(a + b) \arrow[r, "\tilde{\lambda}_{a+b}"] \arrow[d, "\underline{1}_{a,b}"'] & a + b \arrow[dl, equal] \\
a + b &
\end{tikzcd}
$$

Note that in the definition given in [JiPi07] inverses of maps $\underline{a}_{b,c}$ and $\overline{a}_{b,c}$ rather than the maps themselves are considered and diagrams 5.2 and 5.4 are replaced by

5.12

$$
\begin{tikzcd}[column sep=small]
& a.(b+b')+a.(c+c') \arrow[dr, "\underline{a}_{b,b'}+\underline{a}_{c,c'}"] & \\
a.((b+b')+(c+c')) \arrow[ur, "\underline{a}_{b+b',c+c'}"] \arrow[d, dash, "a.\cong"'] & & (a.b+a.b')+(a.c+a.c') \arrow[d, dash, "\cong"] \\
a.((b+c)+(b'+c')) \arrow[dr, "\underline{a}_{b+c,b'+c'}"'] & & (a.b+a.c)+(a.b'+a.c') \\
& a.(b+c)+a.(b'+c') \arrow[ur, "\underline{a}_{b,c}+\underline{a}_{b',c'}"'] &
\end{tikzcd}
$$

and similarly diagrams 5.3 and 5.5 are replaced by

5.13

$$
\begin{tikzcd}[column sep=small]
& (a+a').c+(b+b').c \arrow[dr, "\overline{c}_{a,a'}+\overline{c}_{b,b'}"] & \\
((a+a')+(b+b')).c \arrow[ur, "\overline{c}_{a+a',b+b'}"] \arrow[d, dash, "\cong.c"'] & & (a.c+a'.c)+(b.c+b'.c) \arrow[d, dash, "\cong"] \\
((a+b)+(a'+b')).c \arrow[dr, "\overline{c}_{a+b,a'+b'}"'] & & (a.c+b.c)+(a'.c+b'.c) \\
& (a+b).c+(a'+b').c' \arrow[ur, "\overline{c}_{a,b}+\overline{c}_{a',b'}"'] &
\end{tikzcd}
$$

The definitions here and in [JiPi07] are indeed equivalent (To see this use for instance Lemma [Sch08]-7.1.)

Definition 5.14 *A morphism of categorical ring $\mathcal{A} \to \mathcal{B}$ consists of a functor $H : \mathcal{A} \to \mathcal{B}$ with a symmetric monoidal structure between the symmetric categorical groups*

$$H_+ : (\mathcal{A}, +, 0, ass, r, l, s) \to (\mathcal{B}, +, 0, ass, r, l, s)$$

and a monoidal structure between the monoidal categories

$$H_\times : (\mathcal{A}, ., 1, \tilde{\alpha}, \tilde{\rho}, \tilde{\lambda}) \to (\mathcal{B}, ., 1, \tilde{\alpha}, \tilde{\rho}, \tilde{\lambda})$$

such that the following diagrams

5.15

$$
\begin{tikzcd}[column sep=small]
& H(a).H(b)+H(a).H(b') \arrow[r, "{H_\times}^2_{a,b}+{H_\times}^2_{a,b'}"] & H(a.b)+H(a.b') \arrow[dr, "{H_+}^2_{a.b,a.b'}"] & \\
H(a).(H(b)+H(b')) \arrow[ur, "\underline{H(a)}_{H(b),H(b')}"] \arrow[dr, "H(a).H^2_{+b,b'}"'] & & & H(a.b+a.b') \\
& H(a).H(b+b') \arrow[r, "{H_\times}^2_{a,b+b'}"'] & H(a.(b+b')) \arrow[ur, "H(\underline{a}_{b,b'})"'] &
\end{tikzcd}
$$

and

5.16

$$\begin{tikzcd}[column sep=small]
 & H(a).H(b)+H(a').H(b) \arrow[r, "{H{\times}^2_{a,b}+H{\times}^2_{a',b}}"] & H(a.b)+H(a'.b) \arrow[dr, "{H{+}^2_{a.b,a.b'}}"] & \\
(H(a)+H(a')).H(b) \arrow[ur, "{\overline{H(b)}_{H(a),H(a')}}"] \arrow[dr, "{H{+}^2_{a,a'}.H(b)}"'] & & & H(a.b+a'.b) \\
 & H(a+a').H(b) \arrow[r, "{H{\times}^2_{a+a',b}}"'] & H((a+a').b) \arrow[ur, "{H(\overline{b}_{a,a'})}"'] &
\end{tikzcd}$$

commute for all possible objects involved.

The following is a crucial example of SPC-category.

Proposition 5.17 *The 2-category SPC gets strictly enriched over itself, i.e. it admits a strict enriched structure as follows. The hom map sends any pair $\mathcal{A}$,$\mathcal{B}$ of objects to $[\mathcal{A},\mathcal{B}]$, the composition maps are the $[\mathcal{A},-]_{\mathcal{B},\mathcal{C}} : [\mathcal{B},\mathcal{C}] \to [[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{C}]]$ and the unit arrows $u_{\mathcal{A}}$ are the $v : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$.*

PROOF:See 7.37 in Appendix. ∎

Let us make the following remark about the terminology. If SPC' denotes just for the purpose of this explanation the enriched structure of SPC over itself then for any $\mathcal{A}$ in SPC and any 1-cell $F : \mathcal{B} \to \mathcal{C}$ in SPC, one has that:
- $SPC'(\mathcal{A},-)_{\mathcal{B},\mathcal{C}}$ is $[\mathcal{A},-]_{\mathcal{B},\mathcal{C}} : [\mathcal{B},\mathcal{C}] \to [[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{C}]]$;
- $SPC'(-,\mathcal{C})_{\mathcal{A},\mathcal{B}}$ is $[-,\mathcal{C}]_{\mathcal{A},\mathcal{B}} : [\mathcal{A},\mathcal{B}] \to [[\mathcal{B},\mathcal{C}],[\mathcal{A},\mathcal{C}]]$;
- $SPC'(1,F)$ is $[1,F] : [\mathcal{A},\mathcal{B}] \to [\mathcal{A},\mathcal{C}]$;
- $SPC'(F,1)$ is $[F,1] : [\mathcal{B},\mathcal{C}] \to [\mathcal{A},\mathcal{C}]$.
The first two points results from the definitions. The other two points are the following lemma proved in Appendix 7.39.

Lemma 5.18 *For any $\mathcal{A}$ and any arrows $F : \mathcal{B} \to \mathcal{C}$ and $\tilde{F} : \mathcal{I} \to [\mathcal{B},\mathcal{C}]$ strict with $ev_{\star}(\tilde{F}) = F$ the diagrams in SPC*

$$\begin{tikzcd}
{[\mathcal{C},\mathcal{A}]} \arrow[r, "{[F,\mathcal{A}]}"] \arrow[d, "R'"'] & {[\mathcal{B},\mathcal{A}]} \\
{[\mathcal{C},\mathcal{A}]\otimes\mathcal{I}} \arrow[r, "1\otimes\tilde{F}"'] & {[\mathcal{C},\mathcal{A}]\otimes[\mathcal{B},\mathcal{C}]} \arrow[u, "c"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]} \arrow[r, "{[\mathcal{A},F]}"] \arrow[d, "L'"'] & {[\mathcal{A},\mathcal{C}]} \\
{\mathcal{I}\otimes[\mathcal{A},\mathcal{B}]} \arrow[r, "\tilde{F}\otimes 1"'] & {[\mathcal{B},\mathcal{C}]\otimes[\mathcal{A},\mathcal{B}]} \arrow[u, "c"']
\end{tikzcd}$$

both commute.

Given any SPC-category $\mathcal{A}$, 2-rings are obtained by restriction of $\mathcal{A}$ to anyone of its points. The particular case of the strict enriched structure on SPC yields a strict 2-ring structure on $[\mathcal{A},\mathcal{A}]$ for any Picard category $\mathcal{A}$.

Another important example of 2-ring is provided by the unit $\mathcal{I}$ of SPC.

Proposition 5.19 *The unit $\mathcal{I}$ of SPC admits a strict 2-ring structure with multiplication given by $L_{\mathcal{I}} : \mathcal{I} \otimes \mathcal{I} \to \mathcal{I}$ (or $v : \mathcal{I} \to [\mathcal{I}, \mathcal{I}]$) and unit the identity at $\mathcal{I}$.*

PROOF:See Appendix 7. ∎

6 Modules and their morphisms

Any 2-ring $\mathcal{A}$ yields a category $\mathcal{A}$-mod of $\mathcal{A}$-modules and their morphisms. Formally $\mathcal{A}$-mod is the category of SPC-functors $\mathcal{A} \to SPC$ and SPC-natural transformations between them. In this section we present alternative descriptions of $\mathcal{A}$-modules and their morphisms. In particular we show that the category $\mathcal{A}$-mod is isomorphic to a category of T-algebras and their morphisms for the doctrine $T = \mathcal{A} \otimes -$ over SPC. Eventually we prove in Proposition 6.36 that the category $\mathcal{I} - mod$ of modules over the unit 2-ring $\mathcal{I}$ is equivalent to SPC.

In this section $\mathcal{A}$ stand for a 2-ring with multiplication $c' : \mathcal{A} \to [\mathcal{A}, \mathcal{A}]/\ c : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ with unit $u : \mathcal{I} \to \mathcal{A}$ and coherence 2-cell α'/α (3.11/4.1), ρ/ρ' (4.2/3.12) and λ/λ' (4.3/3.13).

Considering an arbitrary SPC-functor $F : \mathcal{A} \to SPC$, let us write $\mathcal{M}$ for the object $F(\star)$ of SPC image by F of the unique point $\star$ of $\mathcal{A}$, φ' for the arrow unique component $F_{\star,\star} : \mathcal{A} \to [\mathcal{M}, \mathcal{M}]$ of F, β' for the 2-cell ${F'}^2_{\star,\star,\star}$ and γ' for the 2-cell $F^0_\star$. Then one obtains the following first definition of $\mathcal{A}$-modules by rewriting the data 3.16 and 3.17 and Axioms 3.18, 3.19 and 3.20 with these new notations.

A $\mathcal{A}$-module $\mathcal{M} = (\mathcal{M}, \varphi', \beta', \gamma')$ consists of the following data in SPC: an object $\mathcal{M}$, with an arrow $\varphi' : \mathcal{A} \to [\mathcal{M}, \mathcal{M}]$, called its *action*, and two 2-cells β' and γ' as follows

6.1

$$\begin{tikzcd}
& \mathcal{A} \arrow[dl, "\varphi'"'] \arrow[dr, "c'"] & \\
{[\mathcal{M},\mathcal{M}]} \arrow[d, "{[\mathcal{M},-]}"'] & & {[\mathcal{A},\mathcal{A}]} \arrow[d, "{[1,\varphi']}"] \\
{[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[rr, "{[\varphi',1]}"'] \arrow[urr, Rightarrow, "\beta'"] & & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]}
\end{tikzcd}$$

6.2

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u"] \arrow[dr, "v"'] & \mathcal{A} \arrow[d, "\varphi'"] \\
& {[\mathcal{M},\mathcal{M}]} \arrow[u, phantom, "\gamma'"]
\end{tikzcd}$$

and those satisfy the coherence axioms 6.3, 6.4 and 6.5 below.

6.3 *The 2-cells*

$$\begin{tikzcd}
 & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[[\mathcal{M},\mathcal{M}],-]}"] \arrow[d, Rightarrow, "\alpha'_{\mathcal{M},\mathcal{M},\mathcal{M},\mathcal{M}}"] & {[[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]],[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]]} \arrow[d, "{[[\mathcal{M},-],1]}"] & \\
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "c'"'] & {[\mathcal{M},\mathcal{M}]} \arrow[r, "{[\mathcal{M},-]}"] \arrow[ur, "{[\mathcal{M},-]}"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[1,[\mathcal{M},-]]}"] \arrow[d, "{[\varphi',1]}"] \arrow[dll, Rightarrow, "\beta'"'] \arrow[dr, phantom, "="] & {[[\mathcal{M},\mathcal{M}],[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]]} \arrow[d, "{[\varphi',1]}"] & \\
{[\mathcal{A},\mathcal{A}]} \arrow[rr, "{[1,\varphi']}"] \arrow[drr, "{[1,c']}"'] & & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[1,[\mathcal{M},-]]}"] \arrow[d, Rightarrow, "{[1,\beta']}"] & {[\mathcal{A},[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]]} \arrow[r, "{[1,[\varphi',1]]}"] & {[\mathcal{A},[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \\
 & & {[\mathcal{A},[\mathcal{A},\mathcal{A}]]} \arrow[urr, "{[1,[1,\varphi']]}"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[\mathcal{M},\mathcal{M}]} \arrow[r, "{[\mathcal{M},-]}"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[\mathcal{A},-]}"] \arrow[d, "{[\varphi',1]}"] \arrow[ddl, Rightarrow, "\beta'"'] \arrow[dr, phantom, "="] & {[[\mathcal{A},[\mathcal{M},\mathcal{M}]],[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \arrow[r, "{[[\varphi',1],1]}"] \arrow[d, "{[[1,\varphi'],1]}"] & {[[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]],[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \arrow[r, "{[[\mathcal{M},-],1]}"] \arrow[d, Rightarrow, "{[\beta',1]}"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \arrow[d, "{[\varphi',1]}"] \\
 & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[\mathcal{A},-]}"] & {[[\mathcal{A},\mathcal{A}],[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \arrow[rr, "{[c',1]}"'] & & {[\mathcal{A},[\mathcal{A},[\mathcal{M},\mathcal{M}]]]} \\
\mathcal{A} \arrow[uu, "\varphi'"] \arrow[r, "c'"] \arrow[drr, "c'"'] & {[\mathcal{A},\mathcal{A}]} \arrow[u, "{[1,\varphi']}"] \arrow[r, "{[\mathcal{A},-]}"] \arrow[ur, phantom, "="] & {[[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{A}]]} \arrow[u, "{[1,[1,\varphi']]}"] \arrow[rr, "{[c',1]}"] \arrow[d, Rightarrow, "\alpha'"] \arrow[urr, phantom, "="] & & {[\mathcal{A},[\mathcal{A},\mathcal{A}]]} \arrow[u, "{[1,[1,\varphi']]}"'] \\
 & & {[\mathcal{A},\mathcal{A}]} \arrow[urr, "{[1,c']}"'] & &
\end{tikzcd}$$

are equal.

6.4 *The 2-cells in SPC*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\varphi'"] \arrow[d, "c'"'] & & {[\mathcal{M},\mathcal{M}]} \arrow[d, "{[\mathcal{M},-]}"] \arrow[dll, Rightarrow, "\beta'"'] \arrow[dddd, bend left=60, "1"] \\
{[\mathcal{A},\mathcal{A}]} \arrow[dr, "{[1,\varphi']}"] \arrow[dd, "{[u,1]}"'] & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[dl, "{[\varphi',1]}"'] \arrow[dd, "{[v,1]}"] \arrow[dd, phantom, shift left=8ex, "="] \\
 & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[dr, "{[v,1]}"'] \arrow[r, phantom, "\overset{[\gamma',1]}{\Longleftarrow}"] & \\
{[\mathcal{I},\mathcal{A}]} \arrow[rr, "{[1,\varphi']}"] \arrow[d, "ev_\star"'] \arrow[ur, phantom, "="] & & {[\mathcal{I},[\mathcal{M},\mathcal{M}]]} \arrow[d, "ev_\star"] \\
\mathcal{A} \arrow[rr, "\varphi'"'] \arrow[urr, phantom, "="] & & {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "id"] \arrow[d, "c'"'] \arrow[dr, phantom, "\Downarrow\rho'"] & \mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M},\mathcal{M}]} \\
{[\mathcal{A},\mathcal{A}]} \arrow[r, "{[u,1]}"'] & {[\mathcal{I},\mathcal{A}]} \arrow[u, "ev_\star"'] &
\end{tikzcd}$$

are equal.

6.5 *The 2-cells in SPC*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\varphi'"] \arrow[d, "c'^*"'] & & {[\mathcal{M},\mathcal{M}]} \arrow[d, "{[-,\mathcal{M}]}"] \arrow[ll, Rightarrow, shorten=2em, "(\beta')^*"', start anchor=south west, end anchor=north east] \arrow[dddd, bend left=70, "1"] \\
{[\mathcal{A},\mathcal{A}]} \arrow[dr, "{[1,\varphi']}"] \arrow[dd, "{[u,1]}"'] & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[dl, "{[\varphi',1]}"'] \arrow[dd, "{[v,1]}"] \\
 & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[dr, "{[v,1]}"'] \arrow[r, phantom, "\overset{[\gamma',1]}{\Longleftarrow}"] \arrow[dl, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
{[\mathcal{I},\mathcal{A}]} \arrow[rr, "{[1,\varphi']}"] \arrow[d, "ev_\star"'] & & {[\mathcal{I},[\mathcal{M},\mathcal{M}]]} \arrow[d, "ev_\star"] \\
\mathcal{A} \arrow[rr, "\varphi'"'] & & {[\mathcal{M},\mathcal{M}]}
\arrow[from=4-1, to=5-3, phantom, "="]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "id"] \arrow[d, "c'"'] & & \mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M},\mathcal{M}]} \\
{[\mathcal{A},\mathcal{A}]} \arrow[rr, "{[u,1]}"'] & & {[\mathcal{I},\mathcal{A}]} \arrow[u, "ev_\star"']
\arrow[from=1-1, to=2-3, phantom, "\Downarrow\lambda'"]
\end{tikzcd}$$

are equal.

We shall also denote by γ'' for the 2-cell

6.6

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\varphi'^*"] \arrow[d, dash, "id"'] & {[\mathcal{A},\mathcal{M}]} \arrow[d, "{[u,1]}"] \\
\mathcal{M} & {[\mathcal{I},\mathcal{M}]} \arrow[l, "ev_\star"]
\arrow[from=1-1, to=2-2, phantom, "\overset{\gamma''}{\Longrightarrow}"]
\end{tikzcd}$$

that corresponds to γ' via the bijection 7.16/7.17.

Consider now a SPC-natural transformation between presheaves with domain a one point category $\mathcal{A}$

$$(\sigma, \kappa) : F \to G : \mathcal{A} \to SPC.$$

Let us write $\mathcal{M} = (\mathcal{M}, \varphi', \beta', \gamma'')$ and $\mathcal{N} = (\mathcal{N}, \psi', \beta', \gamma'')$ for the two modules corresponding respectively to F and G. The SPC-natural transformation σ has a unique component at the unique object $\star$ of $\mathcal{A}$, which is a strict arrow $\mathcal{I} \to [\mathcal{M}, \mathcal{N}]$ in SPC or equivalently an arrow $H : \mathcal{M} \to \mathcal{N}$. The collection κ consists of a unique 2-cell

6.7

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & {[\mathcal{M},\mathcal{M}]} \arrow[d, "{[1,H]}"] \\
{[\mathcal{N},\mathcal{N}]} \arrow[r, "{[H,1]}"'] \arrow[ur, Rightarrow, shorten=1em] & {[\mathcal{M},\mathcal{N}]}.
\end{tikzcd}$$

which we name δ'. We obtain therefore the following definition of morphism of $\mathcal{A}$-modules by rewriting Axioms 3.31 and 3.32 with these new notations.

A morphism of $\mathcal{A}$-module $(H,\delta') : (\mathcal{M},\varphi',\beta',\gamma') \to (\mathcal{N},\psi',\beta',\gamma')$ consists of an arrow $H : \mathcal{M} \to \mathcal{N}$ in SPC with a 2-cell δ' as in 6.7, those satisfying Axioms 6.8 and 6.9 below.

6.8 *The 2-cells Ξ_1, Ξ_2, Ξ_3, Ξ_4, Ξ_5, Ξ_6, Ξ_7 and Ξ_8 in SPC below satisfy the equality*

$$\Xi_2 \circ \Xi_1 = \Xi_8 \circ \Xi_7 \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ \Xi_3.$$

Ξ_1 *is*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\psi'"] \arrow[d, "c'"'] & {[\mathcal{N},\mathcal{N}]} \arrow[r, "{[\mathcal{N},-]}"] \arrow[d, phantom, "\Downarrow\beta'"] & {[[\mathcal{N},\mathcal{N}],[\mathcal{N},\mathcal{N}]]} \arrow[d, "{[\psi',1]}"] & \\
{[\mathcal{A},\mathcal{A}]} \arrow[rr, "{[1,\psi']}"'] & & {[\mathcal{A},[\mathcal{N},\mathcal{N}]]} \arrow[r, "{[1,[H,1]]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]}
\end{tikzcd}$$

Ξ_2 *is*

$$\begin{tikzcd}
 & & {[\mathcal{A},[\mathcal{N},\mathcal{N}]]} \arrow[dr, "{[1,[H,1]]}"] \arrow[dd, phantom, "\Downarrow{[1,\delta']}"] & \\
\mathcal{A} \arrow[r, "c'"] & {[\mathcal{A},\mathcal{A}]} \arrow[ur, "{[1,\psi']}"] \arrow[dr, "{[1,\varphi']}"'] & & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[ur, "{[1,[1,H]]}"'] &
\end{tikzcd}$$

Ξ_3 *is the identity*

$$\begin{tikzcd}
 & & {[[\mathcal{N},\mathcal{N}],[\mathcal{N},\mathcal{N}]]} \arrow[dr, "{[1,[H,1]]}"] & & \\
\mathcal{A} \arrow[r, "\psi'"] & {[\mathcal{N},\mathcal{N}]} \arrow[ur, "{[\mathcal{N},-]}"] \arrow[dr, "{[\mathcal{M},-]}"'] & = & {[[\mathcal{N},\mathcal{N}],[\mathcal{M},\mathcal{N}]]} \arrow[r, "{[\psi',1]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & & {[[\mathcal{M},\mathcal{N}],[\mathcal{M},\mathcal{N}]]} \arrow[ur, "{[[H,1],1]}"'] & &
\end{tikzcd}$$

Ξ_4 *is*

$$\begin{tikzcd}
 & & & {[[\mathcal{N},\mathcal{N}],[\mathcal{N},\mathcal{N}]]} \arrow[dr, "{[\psi',1]}"] \arrow[dd, phantom, "\Downarrow{[\delta',1]}"] & \\
\mathcal{A} \arrow[r, "\psi'"] & {[\mathcal{N},\mathcal{N}]} \arrow[r, "{[\mathcal{M},-]}"] & {[[\mathcal{M},\mathcal{N}],[\mathcal{M},\mathcal{N}]]} \arrow[ur, "{[[H,1],1]}"] \arrow[dr, "{[[1,H],1]}"'] & & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{N}]]} \arrow[ur, "{[\varphi',1]}"'] &
\end{tikzcd}$$

Ξ_5 *is the identity 2-cell*

$$\begin{tikzcd}
 & & {[[\mathcal{M},\mathcal{N}],[\mathcal{M},\mathcal{N}]]} \arrow[dr, "{[[1,H],1]}"] & & \\
\mathcal{A} \arrow[r, "\psi'"] & {[\mathcal{N},\mathcal{N}]} \arrow[ur, "{[\mathcal{M},-]}"] \arrow[dr, "{[H,1]}"'] \arrow[rr, phantom, "="] & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{N}]]} \arrow[r, "{[\varphi',1]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & & {[\mathcal{M},\mathcal{N}]} \arrow[ur, "{[\mathcal{M},-]}"'] & &
\end{tikzcd}$$

Ξ_6 *is*

$$\begin{tikzcd}
 & {[\mathcal{N},\mathcal{N}]} \arrow[dr, "{[H,1]}"] \arrow[dd, Rightarrow, "\delta'"] & & & \\
\mathcal{A} \arrow[ur, "\psi'"] \arrow[dr, "\varphi'"'] & & {[\mathcal{M},\mathcal{N}]} \arrow[r, "{[\mathcal{M},-]}"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{N}]]} \arrow[r, "{[\varphi',1]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & {[\mathcal{M},\mathcal{M}]} \arrow[ur, "{[1,H]}"'] & & &
\end{tikzcd}$$

Ξ_7 *is the identity 2-cell*

$$\begin{tikzcd}
 & & {[\mathcal{M},\mathcal{N}]} \arrow[dr, "{[\mathcal{M},-]}"] & & \\
\mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M},\mathcal{M}]} \arrow[ur, "{[1,H]}"] \arrow[dr, "{[\mathcal{M},-]}"'] \arrow[rr, phantom, "="] & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{N}]]} \arrow[r, "{[\varphi',1]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]} \\
 & & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[ur, "{[1,[1,H]]}"'] & &
\end{tikzcd}$$

and Ξ_8 *is*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "c'"'] & {[\mathcal{M},\mathcal{M}]} \arrow[r, "{[\mathcal{M},-]}"] \arrow[d, phantom, "\Downarrow\beta'"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[d, "{[\varphi',1]}"] & \\
{[\mathcal{A},\mathcal{A}]} \arrow[rr, "{[1,\varphi']}"'] & & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[r, "{[1,[1,H]]}"] & {[\mathcal{A},[\mathcal{M},\mathcal{N}]]}.
\end{tikzcd}$$

6.9 *The 2-cells in SPC*

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "u"] \arrow[rr, bend left=60, "v"{name=V}] \arrow[rrr, bend left=80, "H"{name=H}] & \mathcal{A} \arrow[r, "\varphi'"'] \arrow[from=V, Rightarrow, shorten=2mm, "\gamma'"] & {[\mathcal{M},\mathcal{M}]} \arrow[r, "{[1,H]}"'] \arrow[from=H, Rightarrow, shorten=2mm, "{u^2}_H"] & {[\mathcal{M},\mathcal{N}]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "v"] \arrow[rr, phantom, bend left=30, "="] \arrow[rrr, bend left=40, "H"] \arrow[d, dash, "id"'] & {[\mathcal{N},\mathcal{N}]} \arrow[rr, "{[H,1]}"] \arrow[drr, Rightarrow, "\delta'"] & & {[\mathcal{M},\mathcal{N}]} \\
\mathcal{I} \arrow[r, "u"'] \arrow[u, phantom, "\Downarrow \gamma'"] & \mathcal{A} \arrow[u, "\psi'"] \arrow[rr, "\varphi'"'] & & {[\mathcal{M},\mathcal{M}]} \arrow[u, "{[1,H]}"']
\end{tikzcd}$$

are equal.

From these first definitions of $\mathcal{A}$-modules and $\mathcal{A}$-module morphisms, one obtains immediately the following simple other definitions involving multilinear maps and multilinear natural transformations.

A $\mathcal{A}$-module $(\mathcal{M}, \underline{\varphi}, \underline{\beta}, \underline{\gamma})$ consists of a bilinear map $\underline{\varphi} : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, which we write as a multiplication $\underline{\varphi}(a, m) = a.m$, with two natural transformations, $\underline{\beta}$ which is trilinear, and $\underline{\gamma}$ which is linear, as follows.

6.10

$$\underline{\beta}_{a_1,a_2,m} : a_1.(a_2.m) \to (a_1.a_2).m$$

lies in $\mathcal{M}$*, for* a_1*,* a_2 *objects of* $\mathcal{A}$ *and* m *object of* $\mathcal{M}$*.*

6.11

$$\underline{\gamma}_m : m \to 1_{\mathcal{A}}.m$$

lies in $\mathcal{M}$ *for* m *object of* $\mathcal{M}$*.*

Those satisfy the coherence conditions 6.12, 6.13 and 6.14 below.

6.12 *For any objects* a_1*,* a_2*,* a_3 *of* $\mathcal{A}$ *and* m *of* $\mathcal{M}$ *the diagram in* $\mathcal{M}$

$$\begin{tikzcd}
a_1.(a_2.(a_3.m))) \arrow[r, "\underline{\beta}_{a_1,a_2,a_3.m}"] \arrow[d, "a_1.\underline{\beta}_{a_2,a_3,m}"'] & (a_1.a_2).(a_3.a) \arrow[dd, "\underline{\beta}_{a_1.a_2,a_3,m}"] \\
a_1.((a_2.a_3).m) \arrow[d, "\underline{\beta}_{a_1,a_2.a_3,m}"'] & \\
(a_1.(a_2.a_3)).m \arrow[r, "\alpha_{a_1,a_2,a_3}.m"'] & ((a_1.a_2).a_3).m
\end{tikzcd}$$

commutes.

6.13 *For any objects* a *of* $\mathcal{A}$ *and* m *of* $\mathcal{M}$ *the diagram in* $\mathcal{M}$

$$\begin{tikzcd}
a.m \arrow[rr, "a.\underline{\gamma}_m"] \arrow[dr, "\rho_a.m"'] & & a.(1_{\mathcal{A}}.m) \arrow[dl, "\underline{\beta}_{a,1_{\mathcal{A}},m}"] \\
& (a.1_{\mathcal{A}}).m &
\end{tikzcd}$$

commutes.

6.14 *For any objects a of $\mathcal{A}$ and m of $\mathcal{M}$ the diagram in $\mathcal{M}$*

$$\begin{tikzcd}
a.m \arrow[rr, "\underline{\gamma}_{a.m}"] \arrow[dr, "\lambda_a.m"'] & & 1_{\mathcal{A}}.(a.m) \arrow[dl, "\underline{\beta}_{1_{\mathcal{A}},a,m}"] \\
& (1_{\mathcal{A}}.a).m &
\end{tikzcd}$$

commutes.

A $\mathcal{A}$-module morphism $(H, \underline{\delta}) : \mathcal{M} \to \mathcal{N}$ consists of an arrow $H : \mathcal{M} \to \mathcal{N}$ in SPC with a bilinear natural transformation

6.15

$$\underline{\delta}_{a,m} : a.H(m) \to H(a.m)$$

which lies in $\mathcal{N}$ for a object of $\mathcal{A}$ and m object of $\mathcal{M}$

which satisfy the Axioms 6.16 and 6.17 below.

6.16 *For any object m of $\mathcal{M}$ the following diagram in $\mathcal{N}$*

$$\begin{tikzcd}
Hm \arrow[rr, "H(\underline{\gamma}_m)"] \arrow[dr, "\underline{\gamma}_{H(m)}"'] & & H(\star.m) \\
& \star.Hm \arrow[ur, "\underline{\delta}_{\star,m}"'] &
\end{tikzcd}$$

commutes.

6.17 *For any objects a_1, a_2 in $\mathcal{A}$ and m in $\mathcal{M}$, the following diagram in $\mathcal{N}$*

$$\begin{tikzcd}
a_1.(a_2.Hm) \arrow[rr, "\underline{\beta}_{a_1,a_2,m}"] \arrow[d, "a_1.\underline{\delta}_{a_2,m}"'] & & (a_1.a_2).Hm \arrow[d, "\underline{\delta}_{a_1.a_2,m}"] \\
a_1.H(a_2.m) \arrow[r, "\underline{\delta}_{a_1,a_2.m}"'] & H(a_1.(a_2.m)) \arrow[r, "H(\underline{\beta}_{a_1,a_2,m})"'] & H((a_1.a_2).m)
\end{tikzcd}$$

commutes.

Note that Axiom 6.17 is obtained from Axiom 6.9 by evaluation at the generator $\star$ since the component in $\star$ of ${u^2}_H$ is an identity. By Remark 7.10 Axioms 6.9 and 6.17 are equivalent.

Eventually we give definitions of $\mathcal{A}$-modules and their morphisms using the tensor in SPC. This will show that in which sense $\mathcal{A}$-modules occur as algebras for the doctrine $\mathcal{A} \otimes -$ over SPC.

A $\mathcal{A}$-module $(\mathcal{M}, \varphi, \beta, \gamma)$ consists of a *strict* arrow $\varphi : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ in SPC in with 2-cells β and γ in SPC as follows

6.18

$$
\begin{tikzcd}
(\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[rr, "A'"] \arrow[d, "c\otimes 1"'] & & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[d, "1\otimes\varphi"] \arrow[dll, Rightarrow, "\beta"'] \\
\mathcal{A}\otimes\mathcal{M} \arrow[dr, "\varphi"'] & & \mathcal{A}\otimes\mathcal{M} \arrow[dl, "\varphi"] \\
& \mathcal{M} &
\end{tikzcd}
$$

and

6.19

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, "L'"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "\overset{\gamma}{\Longrightarrow}"] & \mathcal{I}\otimes\mathcal{M} \arrow[d, "u\otimes 1"] \\
\mathcal{M} & \mathcal{A}\otimes\mathcal{M} \arrow[l, "\varphi"]
\end{tikzcd}
$$

satisfying the coherence conditions 6.20, 6.21 and 6.22 below.

6.20 *The 2-cells*

$$
\begin{tikzcd}
((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \arrow[r, "A'"] \arrow[d, "(c\otimes 1)\otimes 1"'] \arrow[dr, phantom, "="] & (\mathcal{A}\mathcal{A})(\mathcal{A}\mathcal{M}) \arrow[r, dash, "id"] \arrow[d, "c\otimes 1"] & (\mathcal{A}\mathcal{A})(\mathcal{A}\mathcal{M}) \arrow[r, "A'"] \arrow[d, "1\otimes\varphi"] \arrow[dr, phantom, "="] & \mathcal{A}(\mathcal{A}(\mathcal{A}\mathcal{M})) \arrow[d, "1\otimes(1\otimes\varphi)"] \\
(\mathcal{A}\mathcal{A})\mathcal{M} \arrow[r, "A'"] \arrow[d, "c\otimes 1"'] & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[r, phantom, "="] \arrow[d, "1\otimes\varphi"] \arrow[ddl, Rightarrow, "\beta"'] & (\mathcal{A}\mathcal{A})\mathcal{M} \arrow[r, "A'"] \arrow[d, "c\otimes 1"] & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[d, "1\otimes\varphi"] \arrow[ddl, Rightarrow, "\beta"'] \\
\mathcal{A}\mathcal{M} \arrow[d, "\varphi"'] & \mathcal{A}\mathcal{M} \arrow[r, dash, "id"'] \arrow[d, "\varphi"'] \arrow[dr, phantom, "="] & \mathcal{A}\mathcal{M} \arrow[d, "\varphi"'] & \mathcal{A}\mathcal{M} \arrow[d, "\varphi"] \\
\mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \arrow[r, "A'\otimes 1"] \arrow[d, "(c\otimes 1)\otimes 1"'] & ((\mathcal{A}(\mathcal{A}\mathcal{A}))\mathcal{M} \arrow[r, "A'"] \arrow[d, "(1\otimes c)\otimes 1"] \arrow[ddl, Rightarrow, "\alpha\otimes 1"] \arrow[dr, phantom, "="] & \mathcal{A}((\mathcal{A}\mathcal{A})\mathcal{M}) \arrow[r, "1\otimes A'"] \arrow[d, "1\otimes(c\otimes 1)"] & \mathcal{A}(\mathcal{A}(\mathcal{A}\mathcal{M})) \arrow[d, "1\otimes(1\otimes\varphi)"] \arrow[ddl, Rightarrow, "1\otimes\beta"'] \\
(\mathcal{A}\mathcal{A})\mathcal{M} \arrow[d, "c\otimes 1"'] & (\mathcal{A}\mathcal{A})\mathcal{M} \arrow[r, "A'"] \arrow[d, "c\otimes 1"] & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[d, "1\otimes\varphi"] \arrow[ddl, Rightarrow, "\beta"'] & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[d, "1\otimes\varphi"] \\
\mathcal{A}\mathcal{M} \arrow[r, dash, "id"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "="] & \mathcal{A}\mathcal{M} \arrow[d, "\varphi"'] & \mathcal{A}\mathcal{M} \arrow[r, dash, "id"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "="] & \mathcal{A}\mathcal{M} \arrow[d, "\varphi"] \\
\mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M}
\end{tikzcd}
$$

are equal.

6.21 *The 2-cells* $\Xi_1 =$

$$
\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, dash, "id"] \arrow[d, "R'\otimes 1"'] \arrow[dr, phantom, "\Downarrow\rho\otimes 1"] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"] & \mathcal{M} \\
(\mathcal{A}\otimes\mathcal{I})\otimes\mathcal{M} \arrow[r, "(1\otimes u)\otimes 1"'] & (\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[u, "c\otimes 1"'] &
\end{tikzcd}
$$

$\Xi_2 =$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, dash, "id"] \arrow[d, "1\otimes L'"'] \arrow[dr, phantom, "\Downarrow 1\otimes\gamma"] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"] & \mathcal{M} \\
\mathcal{A}\otimes(\mathcal{I}\otimes\mathcal{M}) \arrow[r, "1\otimes(u\otimes 1)"'] & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[u, "1\otimes\varphi"'] &
\end{tikzcd}$$

and
$\Xi_3 =$

$$\begin{tikzcd}
 & (\mathcal{A}\otimes\mathcal{I})\otimes\mathcal{M} \arrow[r, "(1\otimes u)\otimes 1"] \arrow[dd, "A'"] \arrow[ddr, phantom, "="] & (\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[r, "c\otimes 1"] \arrow[dd, "A'"'] & \mathcal{A}\otimes\mathcal{M} \arrow[dr, "\varphi"] & \\
\mathcal{A}\otimes\mathcal{M} \arrow[ur, "R'\otimes 1"] \arrow[dr, "1\otimes L'"'] \arrow[r, phantom, "="] & {} & & & \mathcal{M} \\
 & \mathcal{A}\otimes(\mathcal{I}\otimes\mathcal{M}) \arrow[r, "1\otimes(u\otimes 1)"'] & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "1\otimes\varphi"'] \arrow[uur, Rightarrow, "\beta"] & \mathcal{A}\otimes\mathcal{M} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

satisfy the equality $\Xi_1 = \Xi_3 \circ \Xi_2$.

6.22 *The 2-cells* $\Xi_1 =$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, dash, "id"] \arrow[d, "L'\otimes 1"'] \arrow[dr, phantom, "\Downarrow \lambda\otimes 1"] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"] & \mathcal{M} \\
(\mathcal{I}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[r, "(u\otimes 1)\otimes 1"'] & (\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[u, "c\otimes 1"'] &
\end{tikzcd}$$

$\Xi_2 =$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"] \arrow[d, "L'"'] \arrow[dr, phantom, "="] & \mathcal{M} \arrow[r, dash, "id"] \arrow[d, "L'"'] \arrow[dr, phantom, "\Downarrow\gamma"] & \mathcal{M} \\
\mathcal{I}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "1\otimes\varphi"'] & \mathcal{I}\otimes\mathcal{M} \arrow[r, "u\otimes 1"'] & \mathcal{A}\otimes\mathcal{M} \arrow[u, "\varphi"']
\end{tikzcd}$$

$\Xi_3 =$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[rr, "L'"] \arrow[dr, "L'\otimes 1"'] & \arrow[d, phantom, "\Uparrow\theta"] & \mathcal{I}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "u\otimes 1"] & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "1\otimes\varphi"] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"] & \mathcal{M} \\
 & (\mathcal{I}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[ur, "A'"'] & & & &
\end{tikzcd}$$

where the "canonical" 2-cell θ is defined in Appendix in 7.40 and has image by Rn an identity, and
$\Xi_4 =$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, "L'\otimes 1"] & (\mathcal{I}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[r, "(u\otimes 1)\otimes 1"] \arrow[dd, "A'"'] \arrow[ddr, phantom, "="] & (\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{M} \arrow[r, "\varphi\otimes 1"] \arrow[dd, "A'"'] & \mathcal{A}\otimes\mathcal{M} \arrow[dr, "\varphi"] & \\
 & & & & \mathcal{M} \\
 & \mathcal{I}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "u\otimes 1"'] & \mathcal{A}\otimes(\mathcal{A}\otimes\mathcal{M}) \arrow[r, "1\otimes\varphi"'] \arrow[uur, Rightarrow, "\beta"] & \mathcal{A}\otimes\mathcal{M} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

satisfy the equality $\Xi_1 = \Xi_4 \circ (\Xi_3)^{-1} \circ \Xi_2$.

With the previous definition of $\mathcal{A}$-modules, a morphism of $\mathcal{A}$-modules $(H,\delta) : (\mathcal{M},\varphi,\beta,\gamma) \to (\mathcal{N},\psi,\beta,\gamma)$ consists of an arrow $H : \mathcal{M} \to \mathcal{N}$ with a 2-cell δ in SPC

6.23

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{M} \arrow[r, "1\otimes H"] \arrow[d, "\varphi"'] & \mathcal{A}\otimes\mathcal{N} \arrow[d, "\psi"] \arrow[dl, Rightarrow] \\
\mathcal{M} \arrow[r, "H"'] & \mathcal{N}
\end{tikzcd}$$

that satisfy Axioms 6.24 and 6.25 below.

6.24 *The 2-cells*

$$\begin{tikzcd}
 & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[r, "1\otimes(1\otimes H)"] \arrow[d, phantom, "="] & \mathcal{A}(\mathcal{A}\mathcal{N}) \arrow[dd, "1\otimes\psi"] \arrow[dddl, Rightarrow, "\beta"'] \\
(\mathcal{A}\mathcal{A})\mathcal{M} \arrow[ur, "A'"] \arrow[r, "1\otimes H"] \arrow[d, "c\otimes 1"'] \arrow[dr, phantom, "="] & (\mathcal{A}\mathcal{A})\mathcal{N} \arrow[ur, "A'"'] \arrow[d, "c\otimes 1"] & \\
\mathcal{A}\mathcal{M} \arrow[r, "1\otimes H"] \arrow[d, "\varphi"'] & \mathcal{A}\mathcal{N} \arrow[d, "\psi"] \arrow[dl, Rightarrow, "\delta"'] & \mathcal{A}\mathcal{N} \arrow[dl, "\psi"] \\
\mathcal{M} \arrow[r, "H"'] & \mathcal{N} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\mathcal{A}\mathcal{A})\mathcal{M} \arrow[r, "A'"] \arrow[d, "c\otimes 1"'] & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[r, "1\otimes(1\otimes H)"] \arrow[d, "1\otimes\varphi"] \arrow[ddl, Rightarrow, "\beta"'] & \mathcal{A}(\mathcal{A}\mathcal{N}) \arrow[d, "1\otimes\psi"] \arrow[dl, Rightarrow, "1\otimes\delta"'] \\
\mathcal{A}\mathcal{M} \arrow[d, "\varphi"'] & \mathcal{A}\mathcal{M} \arrow[r, "1\otimes H"] \arrow[d, "\varphi"] & \mathcal{A}\mathcal{N} \arrow[d, "\psi"] \arrow[dl, Rightarrow, "\delta"] \\
\mathcal{M} \arrow[r, dash, "id"'] & \mathcal{M} \arrow[r, "H"'] & \mathcal{N}
\end{tikzcd}$$

are equal.

6.25 *The 2-cells*

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "L'"'] \arrow[rrr, bend left, "1"] & \mathcal{I}\otimes\mathcal{M} \arrow[r, "u\otimes 1"'] \arrow[r, phantom, shift left=6, "\Downarrow\gamma"] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"'] & \mathcal{M} \arrow[r, "H"'] & \mathcal{N}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "L'"] \arrow[rrr, bend left, "1"] & \mathcal{I}\otimes\mathcal{N} \arrow[r, "u\otimes 1"'] \arrow[r, phantom, shift left=6, "\Downarrow\gamma"] & \mathcal{A}\otimes\mathcal{N} \arrow[r, "\psi"] \arrow[dr, Rightarrow, "\delta"] & \mathcal{N} \\
\mathcal{M} \arrow[r, "L'"'] \arrow[u, "H"] \arrow[ur, phantom, "="] & \mathcal{I}\otimes\mathcal{M} \arrow[r, "u\otimes 1"'] \arrow[u, "1\otimes H"] \arrow[ur, phantom, "="] & \mathcal{A}\otimes\mathcal{M} \arrow[r, "\varphi"'] \arrow[u, "1\otimes H"] & \mathcal{M} \arrow[u, "H"']
\end{tikzcd}$$

are equal.

To justify these new definitions let us consider again an arbitrary SPC-functor $F : \mathcal{A} \to SPC$ which defines a $\mathcal{A}$-module $\mathcal{M}$ with multiplication $\varphi' : \mathcal{A} \to [\mathcal{M}, \mathcal{M}]$ and 2-cells β' as in 6.1 and γ' as in 6.2. The multiplication corresponds by adjunction 2.5 to a strict arrow $\varphi : \mathcal{A}\otimes\mathcal{M} \to \mathcal{M}$.

According to the adjunction 2.5, Lemmas 7.21 and [Sch08]-19.6, the map $Rn \circ Rn$ defines a bijection between the sets of 2-cells of the following kinds

$$\begin{tikzcd}
(\mathcal{A}\mathcal{A})\mathcal{M} \arrow[rr, "A'"] \arrow[d, "c\otimes 1"'] & & \mathcal{A}(\mathcal{A}\mathcal{M}) \arrow[d, "1\otimes\varphi"] \arrow[dll, phantom, "\Leftarrow"] \\
\mathcal{A}\mathcal{M} \arrow[dr, "\varphi"'] & & \mathcal{A}\mathcal{M} \arrow[dl, "\varphi"] \\
& \mathcal{M} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "c'"'] & {[\mathcal{M},\mathcal{M}]} \arrow[r, "{[\mathcal{M},-]}"] & {[[\mathcal{M},\mathcal{M}],[\mathcal{M},\mathcal{M}]]} \arrow[d, "{[\varphi',1]}"] \arrow[dll, phantom, "\Leftarrow"] \\
{[\mathcal{A},\mathcal{A}]} & & {[\mathcal{A},[\mathcal{M},\mathcal{M}]]} \arrow[ll, "{[1,\varphi']}"]
\end{tikzcd}$$

and one has a 2-cell β corresponding to $\beta'/{F'}^2_{*,*,*}$ via the above bijection. Note then that β has image by Rn the 2-cell $F^2_{\star,\star,\star}$ which we also write β''. One obtains a 2-cell γ as in 6.19 that is equal to the 2-cell γ'' according Lemma 7.8.

We are going to check that the points 6.26, 6.27, 6.28, 6.29, 6.30 and 6.31 below hold for a SPC-functor F and the related data as above. Since the arrows domains of the 2-cells of the equalities of Axioms 6.20, 6.21 and 6.22 are strict, it will result from the adjunction 2.5 that these axioms are equivalent respectively to Axioms 4.7, 4.8 and 4.9 for the SPC-functor F.

6.26 *The 2-cell*

$$\begin{tikzcd}
(\mathcal{A}\otimes\mathcal{A})\otimes\mathcal{A} \arrow[r, "c\otimes 1"] \arrow[d, "(\varphi'\otimes\varphi')\otimes\varphi'"'] & \mathcal{A}\otimes\mathcal{A} \arrow[r, "c"] \arrow[d, "\varphi'\otimes\varphi'"] & \mathcal{A} \arrow[d, "\varphi'"] \\
{([\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}])\otimes[\mathcal{M},\mathcal{M}]} \arrow[r, "c\otimes 1"'] \arrow[d, "A'"'] \arrow[ur, phantom, "\Rightarrow \beta''\otimes 1"] & {[\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}]} \arrow[r, "c"'] \arrow[ur, phantom, "\Rightarrow \beta''"] \arrow[d, phantom, "="] & {[\mathcal{M},\mathcal{M}]} \arrow[d, "id"] \\
{[\mathcal{M},\mathcal{M}]\otimes([\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}])} \arrow[r, "1\otimes c"'] & {[\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}]} \arrow[r, "c"'] & {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

is the image by Rn of the first of the 2-cells of Axiom 6.20.

PROOF:See 7.41 in Appendix. ∎

6.27 *The 2-cell*

$$\begin{tikzcd}
(\mathcal{A}\mathcal{A})\mathcal{A} \arrow[r, "c\otimes 1"] \arrow[d, "A'"'] & \mathcal{A}\mathcal{A} \arrow[r, "c"] & \mathcal{A} \arrow[d, "id"] \\
\mathcal{A}(\mathcal{A}\mathcal{A}) \arrow[r, "1\otimes c"'] \arrow[d, "\varphi'\otimes(\varphi'\otimes\varphi')"'] \arrow[urr, phantom, "\Rightarrow \alpha"] & \mathcal{A}\mathcal{A} \arrow[r, "c"] \arrow[d, "\varphi'\otimes\varphi'"] & \mathcal{A} \arrow[d, "\varphi'"] \\
{[\mathcal{M},\mathcal{M}]\otimes([\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}])} \arrow[r, "1\otimes c"'] \arrow[ur, phantom, "\Rightarrow 1\otimes\beta''"] & {[\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}]} \arrow[r, "c"'] \arrow[ur, phantom, "\Rightarrow \beta''"] & {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

is the image by Rn of the second 2-cell of Axiom 6.21

PROOF:See 7.42 in Appendix. ∎

6.28 *The 2-cell*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "id"] \arrow[d, "R'"'] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M},\mathcal{M}]} \\
\mathcal{A}\otimes\mathcal{I} \arrow[r, "1\otimes u"'] & \mathcal{A}\otimes\mathcal{A} \arrow[u, "c"'] &
\end{tikzcd}$$

is the image by Rn of the 2-cell Ξ_1 of Axiom 6.21.

PROOF:Immediate. ∎

6.29 *The 2-cell*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "R'"'] \arrow[dr, phantom, "="] & {[\mathcal{M},\mathcal{M}]} \arrow[d, "R'"] \arrow[ddd, bend left=60, "id"] \\
\mathcal{A}\otimes\mathcal{I} \arrow[r, "\varphi'\otimes 1"] \arrow[d, "1\otimes u"'] & {[\mathcal{M},\mathcal{M}]\otimes\mathcal{I}} \arrow[d, "1\otimes v"] \arrow[dl, Rightarrow, "1\otimes\gamma'"'] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "\varphi'\otimes\varphi'"'] \arrow[d, "c"'] & {[\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}]} \arrow[d, "c"] \arrow[dl, Rightarrow, "\beta''"'] \\
\mathcal{A} \arrow[r, "\varphi'"'] & {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

is the image by Rn of the 2-cell $\Xi_3 \circ \Xi_2$ of Axiom 6.21.

PROOF:See 7.44 in Appendix. ∎

6.30 *The 2-cell*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "id"] \arrow[d, "L'"'] \arrow[dr, phantom, "\Downarrow\lambda"] & \mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M},\mathcal{M}]} \\
\mathcal{I}\otimes\mathcal{A} \arrow[r, "u\otimes 1"'] & \mathcal{A}\otimes\mathcal{A} \arrow[u, "c"'] &
\end{tikzcd}$$

is the image by Rn of the 2-cell Ξ_1 of Axiom 6.22.

PROOF:immediate. ∎

6.31 *The 2-cell*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[d, "L'"'] \arrow[dr, phantom, "="] & {[\mathcal{M},\mathcal{M}]} \arrow[d, "L'"] \arrow[ddd, bend left=60, "id"] \\
\mathcal{I}\otimes\mathcal{A} \arrow[r, "1\otimes\varphi'"] \arrow[d, "u\otimes 1"'] & {\mathcal{I}\otimes[\mathcal{M},\mathcal{M}]} \arrow[d, "v\otimes 1"] \arrow[dl, Rightarrow, "\gamma'\otimes 1"'] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "\varphi'\otimes\varphi'"'] \arrow[d, "c"'] & {[\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}]} \arrow[d, "c"] \arrow[dl, Rightarrow, "\beta''"'] \\
\mathcal{A} \arrow[r, "\varphi'"'] & {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

is the image by Rn of the 2-cell $\Xi_4 \circ (\Xi_3)^{-1} \circ \Xi_2$ of Axiom 6.22.

PROOF:See 7.45 in Appendix. ∎

Let us consider now two modules $\mathcal{M}$ and $\mathcal{N}$ with respective multiplications denoted by $\varphi : \mathcal{A}\otimes\mathcal{M} \to \mathcal{M}/\varphi' : \mathcal{A} \to [\mathcal{M},\mathcal{M}]$ and $\psi : \mathcal{A}\otimes\mathcal{N} \to \mathcal{N}/\psi' : \mathcal{A} \to [\mathcal{N},\mathcal{N}]$, with sets of coherence two-cells for both written $\beta/\beta''/\beta'$ and $\gamma/\gamma''/\gamma'$, and an arrow $H : \mathcal{M} \to \mathcal{N}$ in SPC. A 2-cell δ as in 6.23 corresponds by the adjunction 2.5 to a 2-cell δ'. For the above related data the equivalence of Axioms 6.24 and 6.8 results form the two following points 6.32 and 6.33 below whereas the equivalence 6.25 and 6.9 follows from Remark 7.10 and points 6.34 and 6.35.

6.32 *The first of the 2-cell of 6.24 has image by $Rn \circ Rn$ the pasting $\Xi_2 \circ \Xi_1$ of 6.8, it has a strict domain which has a strict image by Rn.*

PROOF:See 7.46 in Appendix. ∎

6.33 *The image by $Rn \circ Rn$ of the second 2-cell of Axiom 6.24 is the pasting $\Xi_7 \circ \Xi_6 \circ \Xi_5 \circ \Xi_4 \circ \Xi_3 \circ \Xi_2 \circ \Xi_1$ of 6.8.*

PROOF:See 7.47 in Appendix. ∎

6.34 *The first 2-cell of Axiom 6.9 has image by $ev_\star$ the first 2-cell of Axiom 6.25 and has a strict domain.*

PROOF:See 7.48 in Appendix. ∎

6.35 *The second 2-cell of Axiom 6.9 has image by $ev_\star$ the second cell of Axiom 6.25.*

PROOF:See 7.49 in Appendix. ∎

For any 2-ring $\mathcal{A}$, a $\mathcal{A}$-module $\mathcal{M}$ is said *strict* when the corresponding SPC-presheaf is strict which is to say that its action $\varphi' : \mathcal{A} \to [\mathcal{M},\mathcal{M}]$ is strict and the 2-cells β' and γ' are identities. A few remarks are in order. Consider any $\mathcal{A}$-module $\mathcal{M}$. If its 2-cell γ' is an identity then certainly its 2-cell γ'' is also an identity. Conversely if the action φ' is strict then for any m in $\mathcal{M}$, $\varphi'^*(m) : \mathcal{A} \to \mathcal{M}$ is strict, and the component $\epsilon_{\varphi'^*(m)\circ u}$ is an identity and from this fact one has that if γ'' is strict then also is γ'. If the 2-cells β are identities then certainly are the 2-cells β'. Conversely if $\mathcal{A}$ is a strict 2-ring and the 2-cells β' are identities then the 2-cells β also are.

Proposition 6.36 *The forgetful functor $\mathcal{I}-Mod \to SPC$ is part of an equivalence of categories. Its equivalence inverse factors as*

$$SPC \xrightarrow{\cong} \mathcal{I}-Mod^s \xrightarrow{inc} \mathcal{I}-Mod$$

where the left functor is an isomorphism between SPC and the full sub-category $\mathcal{I}-Mod^s$ of $\mathcal{I}-Mod$ generated by the strict modules and inc is the inclusion functor.

PROOF:One has a forgetful 2-functor $\mathcal{I}-Mod \to SPC$ and the result follows then from Lemmas 6.37, 6.38 and 6.40 below. ∎

Lemma 6.37 *Any object $\mathcal{A}$ of SPC admits a unique strict $\mathcal{I}$-module structure, its multiplication is given by the arrow $\varphi' = v : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$ (or equivalently $\varphi = L_\mathcal{A} : \mathcal{I}\otimes\mathcal{A} \to \mathcal{A}$).*

PROOF:If $\mathcal{A}$ has a strict $\mathcal{I}$-module structure with multiplication $\varphi' = v : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$ the 2-cell γ'' (6.6) in this case being an identity one has that the composite

$$\mathcal{A} \xrightarrow{\varphi'^*} [\mathcal{I},\mathcal{A}] \xrightarrow{ev_\star} \mathcal{A}$$

in SPC is necessarily the identity at $\mathcal{A}$. Actually there is a unique arrow $f : \mathcal{A} \to [\mathcal{I},\mathcal{A}]$ in SPC with strict images in $[\mathcal{I},\mathcal{A}]$ – or equivalently such that f^* is strict – and such that the composite

$$\mathcal{A} \xrightarrow{f} [\mathcal{I},\mathcal{A}] \xrightarrow{ev_\star} \mathcal{A}$$

is the identity and this arrow is v^*.

Now to establish that $\varphi' = v : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$ gives a strict $\mathcal{I}$-module structure on $\mathcal{A}$, it remains to check that one has in this case an identity 2-cell β' (6.1) which is the commutativity of the external diagram in the pasting below

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "v"] \arrow[dr, "v"] \arrow[dd, "v"'] & {[\mathcal{A},\mathcal{A}]} \arrow[d, "{[\mathcal{A},-]}"] \\
 & {[[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{A}]]} \arrow[d, "{[v,1]}"] \\
{[\mathcal{I},\mathcal{I}]} \arrow[r, "{[1,v]}"'] & {[\mathcal{I},[\mathcal{A},\mathcal{A}]]}
\end{tikzcd}$$

where the top right diagram commutes according to Lemma [Sch08]-18.5and the bottom left also does according to Lemma [Sch08]-18.8. ∎

Lemma 6.38 *For any $\mathcal{I}$-module $\mathcal{A}$ and any arrow $H : \mathcal{A} \to \mathcal{B}$ in SPC there is a unique $\mathcal{I}$-module morphism from $\mathcal{A}$ to the strict $\mathcal{I}$-module structure on the symmetric Picard category $\mathcal{B}$ which underlying map in SPC is H. If the multiplication of $\mathcal{A}$ is given by $\varphi' : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$ this morphism has 2-cell δ' as in 6.7*

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "v"] \arrow[d, "\varphi'"'] & {[\mathcal{B},\mathcal{B}]} \arrow[d, "{[H,1]}"] \arrow[dl, Rightarrow] \\
{[\mathcal{A},\mathcal{A}]} \arrow[r, "{[1,H]}"'] & {[\mathcal{A},\mathcal{B}]},
\end{tikzcd}$$

which is determined by its value in $\star$ by Remark 7.10 and such that

$$(\delta'_\star)_a \;=\; \star.Ha \xrightarrow{id} Ha \xrightarrow{H(\underline{\gamma}_a)} H(\varphi'(\star)(a))\;.$$

PROOF:Let us write $t \times a$ for $\varphi'(t)(a)$ for any objects t of $\mathcal{I}$ and a of $\mathcal{A}$.

The coherence Axiom 6.16 for the pair H and δ', with corresponding bilinear $\underline{\delta}$ as in 6.15, amounts to the commutativity of the diagram in $\mathcal{B}$

$$\begin{tikzcd}
Ha \arrow[rr, "H(\underline{\gamma}_a)"] \arrow[dr, dash, "id"'] & & H(\star \times a) \\
 & \star.Ha. \arrow[ur, "\underline{\delta}_{\star,a}"'] &
\end{tikzcd}$$

That the arrow $H : \mathcal{A} \to \mathcal{B}$ in SPC together with the 2-cell $\underline{\delta}$ defined by the condition above satisfies Axiom 6.17 amounts to the commutativity of the diagram in $\mathcal{B}$

6.39

$$\begin{tikzcd}
t_1.(t_2.Ha) \arrow[rr, dash, "id"] \arrow[d, "t_1.\underline{\delta}_{t_2,a}"'] & & (t_1.t_2).Ha \arrow[d, "\underline{\delta}_{t_1.t_2,a}"] \\
t_1.H(t_2 \times a) \arrow[r, "\underline{\delta}_{t_1,t_2\times a}"'] & H(t_1 \times (t_2 \times a)) \arrow[r, "H(\underline{\beta}_{t_1,t_2,a})"'] & H((t_1.t_2) \times a)
\end{tikzcd}$$

for all objects t_1, t_2 in $\mathcal{I}$ and a in $\mathcal{A}$.

We prove this last point by induction on the structure of the objects t_1 in $\mathcal{I}$ for arbitrary objects t_2 and a.

For $t_1 = I$ diagram 6.39 is the external diagram in the pasting

$$\begin{tikzcd}
I \arrow[rr, dash, "id"] \arrow[dd, dash, "id"'] & & I \arrow[dl, "H^0"'] \arrow[dd, "\underline{\delta}_{I,a}"] \\
& H(I) \arrow[d, "H(\varphi'^0_{t_2\times a})"] \arrow[dr, "H(\varphi'^0_{a})"] & \\
I \arrow[ur, "H^0"] \arrow[r, "\underline{\delta}_{I,t_2\times a}"'] & H(I \times (t_2 \times a)) \arrow[r, "H(\underline{\beta}_{I,t_2,a})"'] & H(I \times a)
\end{tikzcd}$$

in which all diagrams commute and in particular the bottom-right triangle since the natural transformation $\underline{\beta}_{-,t_2,a} : - \times (t_2 \times a) \to (-.t_2) \times a : \mathcal{I} \to \mathcal{A}$ is monoidal.

For $t_1 = \star$ diagram 6.39 is the external diagram in the pasting

$$\begin{tikzcd}
\star.(t_2.Ha) \arrow[r, dash, "id"] \arrow[d, "\star.\underline{\delta}_{t_2,a}"'] & (\star.t_2).Ha \arrow[d, "\underline{\delta}_{\star.t_2,a}"] \\
\star.H(t_2 \times a) \arrow[r, dash, "id"] \arrow[d, dash, "id"'] & H((\star.t_2) \times a) \\
H(t_2 \times a) \arrow[r, "H(\underline{\gamma}_{t_2\times a})"'] & H(\star \times (t_2 \times a)) \arrow[u, "H(\underline{\beta}_{\star,t_2,a})"']
\end{tikzcd}$$

where the bottom diagram commutes by the coherence Axiom 6.14 for the $\mathcal{I}$-module $\mathcal{A}$.

For $t_1 = t_1' \otimes t_1''$ diagram 6.39 is the external diagram in the pasting

$$\begin{tikzcd}
(t_1' \otimes t_1'').(t_2.Ha) \arrow[rr, dash, "id"] \arrow[d, dash, "id"'] & & ((t_1' \otimes t_1'').t_2).Ha \arrow[d, dash, "id"] \\
(t_1'.(t_2.Ha)) \otimes (t_1''.(t_2.Ha)) \arrow[rr, dash, "id"] \arrow[d, "t_1'.\underline{\delta}_{t_2,a} \otimes t_1''.\underline{\delta}_{t_2,a}"'] & & ((t_1'.t_2).Ha) \otimes ((t_1''.t_2).Ha) \arrow[d, "\underline{\delta}_{t_1'.t_2,a} \otimes \underline{\delta}_{t_1''.t_2,a}"] \\
(t_1'.H(t_2 \times a)) \otimes (t_1''.H(t_2 \times a)) \arrow[d, "\underline{\delta}_{t_1',t_2 \times a} \otimes \underline{\delta}_{t_1'',t_2 \times a}"'] & & H((t_1'.t_2) \times a) \otimes H((t_1''.t_2) \times a)) \arrow[d, "H^2_{(t_1'.t_2)\times a,(t_1''.t_2)\times a}"] \\
H(t_1' \times (t_2 \times a)) \otimes H(t_1'' \times (t_2 \times a)) \arrow[d, "H^2_{t_1' \times (t_2 \times a), t_1'' \times (t_2 \times a)}"'] & & H(((t_1'.t_2) \times a) \otimes ((t_1''.t_2) \times a)) \arrow[d, "H((\varphi'^2_{t_1'.t_2,t_1''.t_2})_a)"] \\
H((t_1' \times (t_2 \times a)) \otimes (t_1'' \times (t_2 \times a))) \arrow[urr, "H(\underline{\beta}_{t_1',t_2,a}) \otimes H(\underline{\beta}_{t_1'',t_2,a})"] \arrow[d, "H((\varphi'^2_{t_1',t_1''})_{t_2 \times a})"'] & & H(((t_1'.t_2) \otimes (t_1''.t_2)) \times a) \arrow[d, dash, "id"] \\
H((t_1' \otimes t_1'') \times (t_2 \times a)) \arrow[rr, "H(\underline{\beta}_{t_1' \otimes t_1'',t_2,a})"'] & & H(((t_1' \otimes t_1'').t_2) \times a)
\end{tikzcd}$$

where the middle diagram is commutative if the diagram 6.39 commutes for the values $t_1 = t_1'$ and $t_1 = t_1''$ and the bottom diagram commutes since the natural transformation $\underline{\beta}_{-,t_2,a} : - \times (t_2 \times a) \to (-.t_2) \times a : \mathcal{I} \to \mathcal{A}$ is monoidal.

For $t_1 = t^\bullet$, the diagram 6.39 is

$$\begin{tikzcd}
t^\bullet.(t_2.Ha) \arrow[rr, dash, "id"] \arrow[d, "t^\bullet.\underline{\delta}_{t_2,a}"'] & & (t^\bullet.t_2).Ha \arrow[d, "\underline{\delta}_{t^\bullet.t_2,a}"] \\
t^\bullet.H(t_2 \times a) \arrow[r, "\underline{\delta}_{t^\bullet,t_2 \times a}"'] & H(t^\bullet \times (t_2 \times a)) \arrow[r, "H(\underline{\beta}_{t^\bullet,t_2,a})"'] & H((t^\bullet.t_2) \times a)
\end{tikzcd}$$

Note that according to Lemmas 7.7 and 7.6 the arrow $\delta_{t^\bullet,a}$ is

$$t^\bullet.Ha \;\rule[0.5ex]{2em}{0.4pt}\; (t.Ha)^\bullet \xleftarrow{(\underline{\delta}_{t,a})^\bullet} H(t \times a)^\bullet \;\overset{\cong}{\rule[0.5ex]{2em}{0.4pt}}\; H((t \times a)^\bullet) \;\overset{H(\cong)}{\rule[0.5ex]{2em}{0.4pt}}\; H(t^\bullet \times a).$$

Therefore the left-bottom leg rewrites

1. $t^\bullet.(t_2.Ha) \xrightarrow{t^\bullet.\underline{\delta}_{t_2,a}} t^\bullet.H(t_2 \times a) \xrightarrow{\underline{\delta}_{t^\bullet,t_2 \times a}} H(t^\bullet \times (t_2 \times a)) \xrightarrow{H(\underline{\beta}_{t^\bullet,t_2,a})} H((t^\bullet.t_2) \times a)$
2. $t^\bullet.(t_2.Ha) \;\overset{id}{\rule[0.5ex]{2em}{0.4pt}}\; (t.(t_2.Ha))^\bullet \xleftarrow{(t.\underline{\delta}_{t_2,a})^\bullet} (t.H(t_2 \times a))^\bullet \xleftarrow{(\underline{\delta}_{t,t_2 \times a})^\bullet} H(t \times (t_2 \times a))^\bullet \;\overset{\cong}{\rule[0.5ex]{2em}{0.4pt}}\; H((t \times (t_2 \times a))^\bullet)$ ...
   ... $\overset{H(\cong)}{\rule[0.5ex]{2em}{0.4pt}}\; H(t^\bullet \times (t_2 \times a)) \xrightarrow{H(\underline{\beta}_{t^\bullet,t_2,a})} H((t^\bullet.t_2) \times a)$
3. $t^\bullet.(t_2.Ha) \;\overset{id}{\rule[0.5ex]{2em}{0.4pt}}\; (t.(t_2.Ha))^\bullet \xleftarrow{(t.\underline{\delta}_{t_2,a})^\bullet} (t.H(t_2 \times a))^\bullet \xleftarrow{(\underline{\delta}_{t,t_2 \times a})^\bullet} H(t \times (t_2 \times a))^\bullet \;\overset{\cong}{\rule[0.5ex]{2em}{0.4pt}}\; H((t \times (t_2 \times a))^\bullet)$ ...
   ... $\overset{H(\underline{\beta}_{t,t_2,a}{}^\bullet)}{\rule[0.5ex]{2em}{0.4pt}}\; H(((t.t_2) \times a)^\bullet) \;\overset{H(\cong)}{\rule[0.5ex]{2em}{0.4pt}}\; H((t^\bullet.t_2) \times a)$
4. $t^\bullet.(t_2.Ha) \;\overset{id}{\rule[0.5ex]{2em}{0.4pt}}\; (t.(t_2.Ha))^\bullet \xleftarrow{(t.\underline{\delta}_{t_2,a})^\bullet} (t.H(t_2 \times a))^\bullet \xleftarrow{(\underline{\delta}_{t,t_2 \times a})^\bullet} H(t \times (t_2 \times a))^\bullet \xleftarrow{(H(\underline{\beta}_{t,t_2,a}))^\bullet} H(((t.t_2) \times a))^\bullet$ ...
   ... $\overset{\cong}{\rule[0.5ex]{2em}{0.4pt}}\; H(((t.t_2) \times a)^\bullet) \;\overset{H(\cong)}{\rule[0.5ex]{2em}{0.4pt}}\; H((t^\bullet.t_2) \times a)$

In the derivation above arrows 2. and 3. are equal due to Lemma 7.7 and arrows 3. and 4. are equal due to the naturality of the isomorphism 2.4.

The top-right leg of the diagram above rewrites

1. $t^\bullet.(t_2.Ha) \xrightarrow{id} (t^\bullet.t_2).Ha \xrightarrow{\underline{\delta}_{t^\bullet.t_2,a}} H((t^\bullet.t_2)\times a)$
2. $t^\bullet.(t_2.Ha) \xrightarrow{id} (t^\bullet.t_2).Ha \xrightarrow{id} (t.t_2)^\bullet.Ha \xrightarrow{\underline{\delta}_{(t.t_2)^\bullet,a}} H((t.t_2)^\bullet\times a) \xrightarrow{id} H((t^\bullet.t_2)\times a)$
3. $t^\bullet.(t_2.Ha) \xrightarrow{id} (t.t_2)^\bullet.Ha \xrightarrow{id} ((t.t_2).Ha)^\bullet \xleftarrow{(\underline{\delta}_{t.t_2,a})^\bullet} H((t.t_2)\times a)^\bullet \xrightarrow{\cong} H(((t.t_2)\times a)^\bullet)$ ...
   ... $\xrightarrow{H(\cong)} H((t.t_2)^\bullet\times a) \xrightarrow{id} H((t^\bullet.t_2)\times a)$

Therefore the two legs above are equal when diagram 6.39 commutes for $t_1 = t$. ∎

Lemma 6.40 *For any $\mathcal{I}$-module $\mathcal{A}$ the unique $\mathcal{I}$-module morphism from $\mathcal{A}$ to the strict $\mathcal{I}$-module structure on the symmetric Picard category $\mathcal{A}$ and which underlying map is the identity map $1_\mathcal{A}$ at $\mathcal{A}$ in SPC – given by Lemma 6.38 – is invertible in $\mathcal{I}$-mod.*

PROOF:Let $\varphi' : \mathcal{I} \to [\mathcal{A},\mathcal{A}]$ denote the multiplication of $\mathcal{A}$, and $t\times a$ stand for $\varphi(t)(a)$ for any objects t of $\mathcal{I}$ and a of $\mathcal{A}$. The 2-cell δ' given by 6.38 for the morphism from $\mathcal{A}$ to the strict $\mathcal{I}$-module on $\mathcal{A}$ in SPC, is in this case of the form $\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "v"] \arrow[r, bend right, "\varphi'"'] \arrow[r, phantom, "\Downarrow"] & {[\mathcal{A},\mathcal{A}]} \end{tikzcd}$. Its inverse is therefore of the expected form to be part of a module morphism from the strict $\mathcal{I}$-module on $\mathcal{A}$ to the module $\mathcal{A}$ with multiplication φ'.

Recall that the coherence Axiom 6.16 for the pair $(1_\mathcal{A}, \delta')$ as a morphism from $\mathcal{A}$ with multiplication φ' to the strict $\mathcal{I}$-module on $\mathcal{A}$ is the commutativity of the diagram in $\mathcal{A}$

6.41

$$\begin{tikzcd} a \arrow[rr, "\underline{\gamma}_a"] \arrow[dr, dash, "id"'] & & \star\times a \\ & \star.a \arrow[ur, "\underline{\delta}_{\star,a}"'] & \end{tikzcd}$$

for any object a where the bilinear $\underline{\delta}$ corresponds to δ', whereas Axiom 6.17 amounts to the commutation of the diagram in $\mathcal{A}$

6.42

$$\begin{tikzcd} t_1.(t_2.a) \arrow[rr, dash, "id"] \arrow[d, "t_1.\underline{\delta}_{t_2,a}"'] & & (t_1.t_2).a \arrow[d, "\underline{\delta}_{t_1.t_2,a}"] \\ t_1.(t_2\times a) \arrow[r, "\underline{\delta}_{t_1,t_2\times a}"'] & t_1\times(t_2\times a) \arrow[r, "\underline{\beta}_{t_1,t_2,a}"'] & (t_1.t_2)\times a \end{tikzcd}$$

for all objects t_1, t_2 in $\mathcal{I}$ and a in $\mathcal{A}$.

Axiom 6.16 for the pair $(1_A, \delta'^{-1})$ amounts the commutation for any object a in $\mathcal{A}$ of the diagram

$$\begin{tikzcd}
a \arrow[rr, dash, "id"] \arrow[dr, "\underline{\gamma}_a"'] & & \star.a \\
& \star \times a \arrow[ur, "\underline{\delta}^{-1}_{\star,a}"'] &
\end{tikzcd}$$

which does commute since diagram 6.41 does.

Axiom 6.17 for the pair $(1_{\mathcal{A}}, \delta^{-1})$ is the commutation for any objects t_1, t_2 in $\mathcal{I}$ and a in $\mathcal{A}$ of the diagram

$$\begin{tikzcd}
t_1 \times (t_2 \times a) \arrow[rr, "\beta'_{t_1,t_2,a}"] \arrow[d, "t_1 \times \underline{\delta}^{-1}_{t_2,a}"'] & & (t_1.t_2) \times a \arrow[d, "\underline{\delta}^{-1}_{t_1.t_2,a}"] \\
t_1 \times (t_2.a) \arrow[r, "\underline{\delta}^{-1}_{t_1,t_2.a}"'] & t_1.(t_2.a) \arrow[r, dash, "id"'] & (t_1.t_2).a
\end{tikzcd}$$

which is equivalent to the commutation of the external diagram in the pasting

$$\begin{tikzcd}
& t_1.(t_2.a) \arrow[r, dash, "id"] \arrow[dl, "\underline{\delta}_{t_1,t_2.a}"'] \arrow[d, "t_1.\underline{\delta}_{t_2,a}"] & (t_1.t_2).a \arrow[dd, "\underline{\delta}_{t_1.t_2,a}"] \\
t_1 \times (t_2.a) \arrow[dr, "t_1 \times \underline{\delta}_{t_2,a}"'] & t_1.(t_2 \times a) \arrow[d, "\underline{\delta}_{t_1,t_2 \times a}"] & \\
& t_1 \times (t_2 \times a) \arrow[r, "\underline{\beta}_{t_1,t_2,a}"'] & (t_1.t_2) \times a
\end{tikzcd}$$

in which the left diagram commutes by naturality of $\delta'_{t_1} : v(t_1) \to \varphi'(t_1) : \mathcal{I} \to \mathcal{A}$ and the right diagram is diagram 6.42. ∎

7 Appendix

This section contains various technical developments.

Section 2.

Lemma 7.1 *In any symmetric Picard category the diagram*

$$\begin{tikzcd}
a^\bullet \otimes b^\bullet \arrow[r, "!"] \arrow[d, "s"'] & (b \otimes a)^\bullet \arrow[d, "s^\bullet"] \\
b^\bullet \otimes a^\bullet \arrow[r, "!"'] & (a \otimes b)^\bullet
\end{tikzcd}$$

commutes for any objects a and b.

PROOF:By definition the canonical arrow $a^\bullet \otimes b^\bullet \to (b \otimes a)^\bullet$ is the only arrow f making the diagram

$$\begin{tikzcd}
(a^\bullet \otimes b^\bullet) \otimes (b \otimes a) \arrow[r, "f\otimes 1"] \arrow[d, "\cong"'] & (b \otimes a)^\bullet \otimes (b \otimes a) \arrow[ddd, "j"] \\
a^\bullet \otimes ((b^\bullet \otimes b) \otimes a) \arrow[d, "1\otimes(j\otimes 1)"'] & \\
a^\bullet \otimes (I \otimes a) \arrow[d, "\cong"'] & \\
a^\bullet \otimes a \arrow[r, "j"'] & I
\end{tikzcd}$$

commute (see [Lap83]-p.310). In the following pasting all diagrams commute,

$$\begin{tikzcd}
(a^\bullet \otimes b^\bullet) \otimes (b \otimes a) \arrow[r, "s\otimes 1"] \arrow[d, "1\otimes s"'] & (b^\bullet \otimes a^\bullet) \otimes (b \otimes a) \arrow[r, "!\otimes 1"] \arrow[d, "1\otimes s"] & (a \otimes b)^\bullet \otimes (b \otimes a) \arrow[r, "s^\bullet\otimes 1"] \arrow[d, "1\otimes s"] & (b \otimes a)^\bullet \otimes (b \otimes a) \arrow[d, "j"] \\
(a^\bullet \otimes b^\bullet) \otimes (a \otimes b) \arrow[r, "s\otimes 1"] & (b^\bullet \otimes a^\bullet) \otimes (a \otimes b) \arrow[r, "!\otimes 1"] \arrow[d, "\cong"] & (a \otimes b)^\bullet \otimes (a \otimes b) \arrow[r, "j"] & I \\
& b^\bullet \otimes ((a^\bullet \otimes a) \otimes b) \arrow[r, "1\otimes(j\otimes 1)"'] & b^\bullet \otimes (I \otimes b) \arrow[r, "1\otimes r"'] & b^\bullet \otimes b \arrow[u, "j"]
\end{tikzcd}$$

and by the coherence theorem for symmetric monoidal categories the two left legs of the two diagrams above are equal. ∎

7.2 *Proof of Lemma 2.2.*

PROOF:That the functor $inv : \mathcal{A} \to \mathcal{A}$ is monoidal result from points 7.3 that it is symmetric amounts to Lemma 7.1.

7.3 *In any symmetric Picard category the diagram*

$$\begin{tikzcd}
a^\bullet \otimes (b^\bullet \otimes c^\bullet) \arrow[r, "ass"] \arrow[d, "1\otimes s"'] & (a^\bullet \otimes b^\bullet) \otimes c^\bullet \arrow[d, "s\otimes 1"] \\
a^\bullet \otimes (c^\bullet \otimes b^\bullet) \arrow[d, "1\otimes !"'] & (b^\bullet \otimes a^\bullet) \otimes c^\bullet \arrow[d, "!\otimes 1"] \\
a^\bullet \otimes (b \otimes c)^\bullet \arrow[d, "s"'] & (a \otimes b)^\bullet \otimes c^\bullet \arrow[d, "s"] \\
(b \otimes c)^\bullet \otimes a^\bullet \arrow[d, "!"'] & c^\bullet \otimes (a \otimes b)^\bullet \arrow[d, "!"] \\
(a \otimes (b \otimes c))^\bullet \arrow[r, "ass^\bullet"'] & ((a \otimes b) \otimes c)^\bullet
\end{tikzcd}$$

commutes for any objects a, b and c.

PROOF:By the naturality of s, the lemma is equivalent to the commutation of the external diagram in the pasting

$$\begin{tikzcd}
a^\bullet \otimes (b^\bullet \otimes c^\bullet) \arrow[r, "ass"] \arrow[d, "1\otimes s"'] & (a^\bullet \otimes b^\bullet) \otimes c^\bullet \arrow[d, "s\otimes 1"] \\
a^\bullet \otimes (c^\bullet \otimes b^\bullet) \arrow[d, "s"'] & (b^\bullet \otimes a^\bullet) \otimes c^\bullet \arrow[d, "s"] \\
(c^\bullet \otimes b^\bullet) \otimes a^\bullet \arrow[r, "ass"] \arrow[d, "!\otimes 1"'] & c^\bullet \otimes (b^\bullet \otimes a^\bullet) \arrow[d, "1\otimes !"] \\
(b \otimes c)^\bullet \otimes a^\bullet \arrow[d, "!"'] & c^\bullet \otimes (a \otimes b)^\bullet \arrow[d, "!"] \\
(a \otimes (b \otimes c))^\bullet \arrow[r, "ass^\bullet"'] & ((a \otimes b) \otimes c)^\bullet
\end{tikzcd}$$

where the top diagram commutes according to the coherence for the symmetric monoidal structure and the bottom theorem commutes according to the coherence theorem for the group structure. ∎

7.4 *Proof of Lemma 2.3.*

PROOF:The collection j is natural by definition of the functor *inv*. That it is monoidal amounts to the commutation of the diagram

$$\begin{tikzcd}
I \arrow[r, "j_{a\otimes b}"] \arrow[ddd, dash, "\cong"'] & (a \otimes b)^\bullet \otimes (a \otimes b) \arrow[d, "!\otimes 1"] \\
& (b^\bullet \otimes a^\bullet) \otimes (a \otimes b) \arrow[d, "s\otimes 1"] \\
& (a^\bullet \otimes b^\bullet) \otimes (a \otimes b) \arrow[d, dash, "\cong"] \\
I \otimes I \arrow[r, "j_a \otimes j_b"'] & (a^\bullet \otimes a) \otimes (b^\bullet \otimes b)
\end{tikzcd}$$

for any objects a and b of $\mathcal{A}$, which holds by definition of the arrow $! : I : (a \otimes b)^\bullet \to b^\bullet \otimes a^\bullet$ see [Lap83] p.310. ∎

7.5 *Definition of the isomorphism 2.4.*

The isomorphism 2.4 is defined pointwise in a as the only arrow in $\mathcal{B}$ making the diagram

$$
\begin{tikzcd}
I \arrow[r, "j_{Fa}"] \arrow[d, "F^0"'] & (Fa)^\bullet \otimes Fa \arrow[d, dash, "\cong\otimes 1"] \\
FI \arrow[dr, "F(j_a)"'] & F(a^\bullet) \otimes Fa \arrow[d, "F^2_{a^\bullet,a}"] \\
 & F(a^\bullet \otimes a)
\end{tikzcd}
$$

commute.

Lemma 7.6 *Given arrows* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} C$ *in SPC the diagram in* $\mathcal{C}$

$$
\begin{tikzcd}
(FG(a))^\bullet \arrow[r, dash, "\cong"] \arrow[rr, dash, bend right, "\cong"'] & F(G(a)^\bullet) \arrow[r, dash, "F(\cong)"] & FG(a^\bullet)
\end{tikzcd}
$$

where all the $\cong$ *are of type 2.4, is commutative.*

PROOF:Consider the pasting of commutative diagrams below where all the $\cong$ are of type 2.4

$$
\begin{tikzcd}
I \arrow[rr, "j_{FGa}"] \arrow[dd, "F^0"] \arrow[ddd, bend right, "(FG)^0"'] & & (FGa)^\bullet \otimes FGa \arrow[dl, "\cong\otimes 1"'] \arrow[dr, "\cong\otimes 1"] & \\
 & F(Ga)^\bullet \otimes FGa \arrow[rr, "F(\cong)\otimes 1"'] \arrow[d, "F^2_{(Ga)^\bullet,Ga}"'] & & FG(a^\bullet) \otimes FGa \arrow[d, "F^2_{G(a^\bullet),Ga}"'] \arrow[dd, bend left, "(FG)^2_{a^\bullet,a}"] \\
FI \arrow[r, "Fj_{Ga}"'] \arrow[d, "F(G^0)"] & F((Ga)^\bullet \otimes Ga) \arrow[rr, "F(\cong\otimes 1)"'] & & F(G(a^\bullet) \otimes Ga) \arrow[d, "F(G^2_{a^\bullet,a})"'] \\
FGI \arrow[rrr, "FGj_a"'] & & & FG(a^\bullet \otimes a)
\end{tikzcd}
$$

∎

Lemma 7.7 *For any monoidal transformation* $\sigma : F \to G : \mathcal{A} \to \mathcal{B}$ *where* $\mathcal{A}$ *and* $\mathcal{B}$ *are objects of SPC, for any object* a *of* $\mathcal{A}$*, the diagram in* $\mathcal{B}$

$$
\begin{tikzcd}
(Fa)^\bullet \arrow[d, "\cong"'] & (Ga)^\bullet \arrow[l, "(\sigma_a)^\bullet"'] \arrow[d, "\cong"] \\
F(a^\bullet) \arrow[r, "\sigma_{a^\bullet}"'] & G(a^\bullet)
\end{tikzcd}
$$

where the $\cong$ *denote isomorphisms of type 2.4, is commutative.*

PROOF:Consider the pasting of diagrams in $\mathcal{B}$

$$\begin{tikzcd}
I \arrow[r, "j_{Fa}"] \arrow[drr, "id"] \arrow[dd, "F^0"'] & (Fa)^\bullet \otimes Fa \arrow[d, dotted, "\cong\otimes 1"] \arrow[drr, dotted, "{[(\sigma_a)^\bullet]^{-1}\otimes\sigma_a}"] & & \\
 & F(a^\bullet)\otimes Fa \arrow[d, "F^2_{a^\bullet,a}"'] \arrow[drr, dotted, "\sigma_{a^\bullet}\otimes\sigma_a"] & I \arrow[r, "j_{Ga}"] \arrow[dd, "G^0"'] & (Ga)^\bullet\otimes Ga \arrow[d, dotted, "\cong\otimes 1"] \\
FI \arrow[r, "Fj_a"] \arrow[drr, "\sigma_I"'] & F(a^\bullet\otimes a) \arrow[drr, "\sigma_{a^\bullet\otimes a}"] & & G(a^\bullet)\otimes Ga \arrow[d, "G^2_{a^\bullet,a}"] \\
 & & GI \arrow[r, "Gj_a"'] & G(a^\bullet\otimes a)
\end{tikzcd}$$

where the $\cong$ denote isomorphisms of type 2.4. All diagrams above commute apart from the one consisting of the four dotted arrows. Since all arrows are invertible this last diagram commutes. The result follows since tensoring with Fa is an equivalence $\mathcal{B}\to\mathcal{B}$. ∎

A few computational lemmas for ***SPC*****.**
We present here a couple of results not stated in [Sch08].

Lemma 7.8 *The 2-cells in SPC*

$$\mathcal{A} \xrightarrow{L'} \mathcal{I}\otimes\mathcal{A} \xrightarrow{U\otimes 1} \mathcal{B}\otimes\mathcal{A} \xRightarrow{\tau} \mathcal{C}$$

and

$$\mathcal{A} \xRightarrow{Rn(\tau)^*} [\mathcal{B},\mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I},\mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$$

are equal for any 2-cell $\tau:\mathcal{B}\otimes\mathcal{A}\to\mathcal{C}$ and $U:\mathcal{I}\to\mathcal{B}$.

PROOF:The 2-cell $\tau\circ(U\otimes 1)\circ L'$ above rewrites successively

1. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{I},\mathcal{I}\otimes\mathcal{A}] \xrightarrow{ev_\star} \mathcal{I}\otimes\mathcal{A} \xrightarrow{U\otimes 1} \mathcal{B}\otimes\mathcal{A} \xRightarrow{\tau} \mathcal{C}$
2. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{I},\mathcal{I}\otimes\mathcal{A}] \xrightarrow{[1,U\otimes 1]} [\mathcal{I},\mathcal{B}\otimes\mathcal{A}] \xRightarrow{[1,\tau]} [\mathcal{I},\mathcal{B}\otimes\mathcal{A}] \xRightarrow{ev_\star} \mathcal{C}$
3. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B},\mathcal{B}\otimes\mathcal{A}] \xrightarrow{[U,1]} [\mathcal{I},\mathcal{B}\otimes\mathcal{A}] \xRightarrow{[1,\tau]} [\mathcal{I},\mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$
4. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B},\mathcal{B}\otimes\mathcal{A}] \xRightarrow{[1,\tau]} [\mathcal{B},\mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I},\mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$
5. $\mathcal{A} \xRightarrow{(Rn(\tau))^*} [\mathcal{B},\mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I},\mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}.$

∎

Lemma 7.9 *The 2-cells in SPC*

$$\mathcal{A} \xrightarrow{R'} \mathcal{A}\otimes\mathcal{I} \xrightarrow{1\otimes U} \mathcal{A}\otimes\mathcal{B} \xRightarrow{\tau} \mathcal{C}$$

and

$$\mathcal{A} \xRightarrow{Rn(\tau)} [\mathcal{B},\mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I},\mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$$

are equal for any 2-cell $\tau:\mathcal{A}\otimes\mathcal{B}\to\mathcal{C}$ and any arrow $U:\mathcal{I}\to\mathcal{B}$.

PROOF:The 2-cell $\tau \circ (1 \otimes U) \circ R'$ above rewrites successively

1. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \xrightarrow{ev_\star} \mathcal{A} \otimes \mathcal{I} \xrightarrow{1 \otimes U} \mathcal{A} \otimes \mathcal{B} \xRightarrow{\tau} \mathcal{C}$
2. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \xrightarrow{[1, 1 \otimes U]} [\mathcal{I}, \mathcal{A} \otimes \mathcal{B}] \xRightarrow{[1,\tau]} [\mathcal{I}, \mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$
3. $\mathcal{A} \xrightarrow{\eta} [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[U,1]} [\mathcal{I}, \mathcal{A} \otimes \mathcal{B}] \xRightarrow{[1,\tau]} [\mathcal{I}, \mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$
4. $\mathcal{A} \xrightarrow{\eta} [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \xRightarrow{[1,\tau]} [\mathcal{A}, \mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I}, \mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$
5. $\mathcal{A} \xRightarrow{Rn(\tau)} [\mathcal{B}, \mathcal{C}] \xrightarrow{[U,1]} [\mathcal{I}, \mathcal{C}] \xrightarrow{ev_\star} \mathcal{C}$

where in the above derivation arrows 1. and 2. are equal according to Corollary [Sch08]-11.2. ∎

Remark 7.10 *Any 2-cell $\sigma : F \to G : \mathcal{I} \to \mathcal{A}$ with F* strict *in SPC is fully determined by its component in $\star$ since F being strict the component at F of counit of the adjunction $\epsilon_F : v^* \circ ev_\star(F) \to F$ is an identity and by naturality of ϵ one has the commutation of*

$$\begin{tikzcd}
v^*(F(\star)) \arrow[r, "v^*(\sigma_\star)"] \arrow[d, dash, "id"'] & v^*(G(\star)) \arrow[d, "\epsilon^G"] \\
F \arrow[r, "\sigma"'] & G
\end{tikzcd}$$

in SPC.

Lemma 7.11 *For any* strict *arrow $F : \mathcal{I} \to \mathcal{A}$ the diagram in SPC*

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "v"] \arrow[d, "F"'] & {[\mathcal{A}, \mathcal{A}]} \arrow[d, "{[F,1]}"] \\
\mathcal{A} & {[\mathcal{I}, \mathcal{A}]} \arrow[l, "ev_\star"]
\end{tikzcd}$$

commutes.

PROOF:The arrow

$$\mathcal{I} \xrightarrow{v} [\mathcal{A}, \mathcal{A}] \xrightarrow{[F,1]} [\mathcal{I}, \mathcal{A}]$$

has dual

$$\mathcal{I} \xrightarrow{F} \mathcal{A} \xrightarrow{v^*} [\mathcal{I}, \mathcal{A}].$$

Since the arrow F is strict it is equal to $v^* \circ ev_\star(F)$ and the result follows then from Lemma [Sch08]-11.9. ∎

Remark 7.12 *Since the units of the adjunctions $v^* \dashv ev_\star$ are identities in SPC, one has a bijection for any $\mathcal{A}$ and $\mathcal{B}$ between sets of 2-cells in SPC of the following kind*

7.13

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, "v^*"] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{A} \arrow[ur, "F"] \arrow[rr, "G"'] & {} & {[\mathcal{I}, \mathcal{B}]}
\end{tikzcd}$$

and

7.14

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F"] \arrow[dr, "G"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{B} \\
& {[\mathcal{I},\mathcal{A}]} \arrow[ur, "ev_\star"'] &
\end{tikzcd}$$

sending any 2-cell Ξ *of the type 7.13 to*

$$\begin{tikzcd}
& \mathcal{B} \arrow[dr, "v^*"] \arrow[d, phantom, "\Downarrow\Xi"] & & \\
\mathcal{A} \arrow[ur, "F"] \arrow[rr, "G"'] & {} & {[\mathcal{I},\mathcal{B}]} \arrow[r, "ev_\star"] & \mathcal{B}
\end{tikzcd}$$

with inverse sending any 2-cell Ξ' *of type 7.14 to*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F"] \arrow[dr, "G"'] & \arrow[d, phantom, "\Downarrow\Xi'"] & \mathcal{B} \arrow[dr, "v^*"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
& {[\mathcal{I},\mathcal{A}]} \arrow[ur, "ev_\star"'] \arrow[rr, no head, "id"'] & {} & \mathcal{A}
\end{tikzcd}$$

We shall describe for any arrow $\varphi : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ and $U : \mathcal{I} \to \mathcal{A}$ of SPC some bijections between sets of 2-cells of the following kind:

7.15

$$\begin{tikzcd}
\mathcal{I}\mathcal{M} \arrow[r, "U\otimes 1"] \arrow[dr, "L"'] & \mathcal{A}\mathcal{M} \arrow[d, "\varphi"] \arrow[dl, phantom, "\Uparrow"] \\
& \mathcal{M}
\end{tikzcd}$$

7.16

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "U"] \arrow[dr, "v"'] & \mathcal{A} \arrow[d, "Rn(\varphi)"] \arrow[dl, phantom, "\Uparrow"] \\
& {[\mathcal{M},\mathcal{M}]}
\end{tikzcd}$$

7.17

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "(Rn(\varphi))^*"] \arrow[dr, "v^*"'] & {[\mathcal{A},\mathcal{M}]} \arrow[d, "{[U,1]}"] \arrow[dl, phantom, "\Uparrow"] \\
& {[\mathcal{I},\mathcal{M}]}
\end{tikzcd}$$

7.18

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "(Rn(\varphi))^*"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "\Rightarrow"] & {[\mathcal{A}, \mathcal{M}]} \arrow[d, "{[U,1]}"] \\
\mathcal{M} & {[\mathcal{I}, \mathcal{M}]} \arrow[l, "ev_\star"]
\end{tikzcd}$$

and

7.19

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "L'"] \arrow[d, dash, "id"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{I}\mathcal{M} \arrow[d, "U\otimes 1"] \\
\mathcal{M} & \mathcal{A}\mathcal{M} \arrow[l, "\varphi"]
\end{tikzcd}$$

as follows.

- Since L is the image by En of $v : \mathcal{I} \to [\mathcal{M}, \mathcal{M}]$ and the image by Rn of

$$\mathcal{I}\mathcal{M} \xrightarrow{U\otimes 1} \mathcal{A}\mathcal{M} \xrightarrow{\varphi} \mathcal{M}$$

is

$$\mathcal{I} \xrightarrow{U} \mathcal{A} \xrightarrow{Rn(\varphi)} [\mathcal{M}, \mathcal{M}]$$

the maps Rn/En define the bijection (and its inverse) between sets of 2-cells 7.15 and 7.16.

- 2-cells 7.16 and 7.17 correspond by duality.

- The bijection between sets of 2-cells 7.17 and 7.18 sends any 2-cell $\Xi : v^* \to [j, 1] \circ (Rn(\varphi))^*$ to

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "(Rn(\varphi))^*"] \arrow[dr, "v^*"'] \arrow[dr, phantom, bend left=20, "\Xi"] & {[\mathcal{A}, \mathcal{M}]} \arrow[d, "{[U,1]}"] & \\
& {[\mathcal{I}, \mathcal{M}]} \arrow[r, "ev_\star"'] & \mathcal{M}
\end{tikzcd}$$

which is as expected a 2-cell

$$id \to ev_\star \circ [U, 1] \circ (Rn(\varphi))^*$$

according to the adjunction 2.21. Its inverse sends any 2-cell $\Xi : 1 \to ev_\star \circ [U, 1] \circ (Rn(\varphi))^*$ to the pasting

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "id"] \arrow[d, "(Rn(\varphi))^*"'] \arrow[dr, phantom, "\Downarrow\Xi"] & \mathcal{M} \arrow[r, "v^*"] \arrow[dr, phantom, "\Downarrow\epsilon"] & {[\mathcal{I}, \mathcal{M}]} \\
{[\mathcal{A}, \mathcal{M}]} \arrow[r, "{[U,1]}"'] & {[\mathcal{I}, \mathcal{M}]} \arrow[u, "ev_\star"] \arrow[ur, "id"'] &
\end{tikzcd}$$

- Eventually 2-cells 7.18 and 7.19 are the same since their codomains arrows respectively $ev_\star \circ [U, 1] \circ Rn(\varphi)^*$ and $\varphi \circ (U \otimes 1) \circ L'$ are equal according to Lemma 7.8.

Lemma 7.20 *The above bijection between sets of 2-cells 7.15/7.19 sends any* $\Xi : L \to \varphi \circ (U \otimes 1) : \mathcal{IM} \to \mathcal{M}$ *to the pasting*

$$\begin{tikzcd}
\mathcal{IM} \arrow[r, "U\otimes 1"] \arrow[dr, "L"] & \mathcal{AM} \arrow[d, "\varphi"] \arrow[dl, phantom, "\Xi\ \Nearrow", near start] \\
\mathcal{M} \arrow[u, "L'"] \arrow[r, "id"'] \arrow[ur, phantom, "=", near start] & \mathcal{M}
\end{tikzcd}$$

PROOF:According to Lemma 7.8, the above 2-cell $\mathcal{A} \xrightarrow{L'} \mathcal{IA} \overset{\Xi}{\Longrightarrow} \mathcal{A}$ is

$$\mathcal{A} \overset{Rn(\Xi)^*}{\Longrightarrow} [\mathcal{I}, \mathcal{A}] \xrightarrow{ev_\star} \mathcal{A}.$$

Consider then the image of the above 2-cell by the bijection 7.18 $\to$ 7.17, it is

$$\begin{tikzcd}
& {[\mathcal{A},\mathcal{A}]} \arrow[dr, "{[U,1]}"] \arrow[d, phantom, "\Downarrow (Rn(\Xi))^*"] & & \mathcal{A} \arrow[dr, "v^*"] \arrow[d, phantom, "\Downarrow \epsilon"] & \\
\mathcal{A} \arrow[ur, "Rn(\varphi)"] \arrow[rr, "v^*"'] & {} & {[\mathcal{I},\mathcal{A}]} \arrow[ur, "ev_\star"] \arrow[rr, dash, "id"'] & {} & {[\mathcal{I},\mathcal{A}]}
\end{tikzcd}$$

which is just $(Rn(\bar{\lambda}))^* : \mathcal{A} \to [\mathcal{I}, \mathcal{A}]$ since $\mathcal{A} \xrightarrow{v^*} [\mathcal{I}, \mathcal{A}] \overset{\epsilon}{\Longrightarrow} \mathcal{A}$ is an identity 2-cell, which dual has image by En the 2-cell $\bar{\lambda}$. ∎

Easy computation also gives the following.

Remark 7.21 *For any arrows* $F : \mathcal{A} \to [\mathcal{B}, \mathcal{X}]$ *and* $G : \mathcal{X} \to [\mathcal{C}, \mathcal{D}]$ *of SPC the arrow*

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{En(F)\otimes 1} \mathcal{X} \otimes \mathcal{C} \xrightarrow{En(G)} \mathcal{D}$$

has image by Rn

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{En(F)} \mathcal{X} \xrightarrow{G} [\mathcal{C}, \mathcal{D}]$$

and has image by $Rn \circ Rn$ *the arrow*

$$\mathcal{A} \xrightarrow{F} [\mathcal{B}, \mathcal{X}] \xrightarrow{[1,G]} [\mathcal{B}, [\mathcal{C}, \mathcal{D}]].$$

Sections 3 and 4.

7.22 *Proof of Equality (I) in second pasting of Axiom 3.15.*

PROOF:To check the equality (I) of arrows, consider the derivation of equal composite arrows in SPC for any arrows $d : \mathcal{A} \to [\mathcal{A}, \mathcal{A}]$ and $j : \mathcal{I} \to \mathcal{A}$.

1. $[\mathcal{A}, \mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d,1]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{[j,1]} [\mathcal{I}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{ev_\star} [\mathcal{A}, \mathcal{B}]$
2. $[\mathcal{A}, \mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d,1]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{ev_{j(\star)}} [\mathcal{A}, \mathcal{B}]$
3. $[\mathcal{A}, \mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{B}]] \xrightarrow{[d^*,1]} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{D} [\mathcal{A}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{ev_{j(\star)}} [\mathcal{A}, \mathcal{B}]$

4. $[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{B}]] \xrightarrow{[d^*,1]} [\mathcal{A},[\mathcal{A},\mathcal{B}]] \xrightarrow{[1,ev_{j(\star)}]} [\mathcal{A},\mathcal{B}]$

5. $[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{B}]] \xrightarrow{[1,ev_{j(\star)}]} [\mathcal{A},[\mathcal{A},\mathcal{B}]] \xrightarrow{[d^*,1]} [\mathcal{A},\mathcal{B}]$

6. $[\mathcal{A},\mathcal{B}] \xrightarrow{[ev_{j(\star)},1]} [[\mathcal{A},\mathcal{A}],\mathcal{B}] \xrightarrow{[d^*,1]} [\mathcal{A},\mathcal{B}]$

7. $[\mathcal{A},\mathcal{B}] \xrightarrow{[ev_\star,1]} [[\mathcal{I},\mathcal{A}],\mathcal{B}] \xrightarrow{[[j,1],1]} [[\mathcal{A},\mathcal{A}],\mathcal{B}] \xrightarrow{[d^*,1]} [\mathcal{A},\mathcal{B}]$

where the equalities between arrows stand for the following reasons:
- 1. and 2. : by the naturality in $\mathcal{A}$ of the collection of arrows $q_{\mathcal{A}} : \mathcal{A} \to [[\mathcal{A},\mathcal{C}],\mathcal{C}]$
- 2. and 3. : Lemma [Sch08]-10.9;
- 3. and 4. : Lemma [Sch08]-11.9;
- 5. and 6. : Lemma [Sch08]-11.3;
- 6. and 7. : by the naturality of the collections of arrows q;

∎

7.23 *Definition of the 2-cell $c^1{}_{f:x'\to x,y,z}$.*

The domain of this 2-cell rewrites

1. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}_{x',x},-]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],[\mathcal{A}_{x',x},\mathcal{A}_{x',z}]] \xrightarrow{[\mathcal{A}(x',-),1]} [\mathcal{A}_{x,y},[\mathcal{A}_{x',x},\mathcal{A}_{x',z}]] \xrightarrow{[1,[f,1]]} [\mathcal{A}_{x,y},[\mathcal{I},\mathcal{A}_{x',z}]] \cdots$
$\cdots \xrightarrow{[1,ev_\star]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$

2. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}_{x',x},-]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],[\mathcal{A}_{x',x},\mathcal{A}_{x',z}]] \xrightarrow{[1,[f,1]]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],[\mathcal{I},\mathcal{A}_{x',z}]] \xrightarrow{[1,ev_\star]} \cdots$
$\cdots [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}(x',-),1]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$

3. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{I},-]} [[\mathcal{I},\mathcal{A}_{x',y}],[\mathcal{I},\mathcal{A}_{x',z}]] \xrightarrow{[[f,1],1]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],[\mathcal{I},\mathcal{A}_{x',z}]] \xrightarrow{[1,ev_\star]} \cdots$
$\cdots [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}(x',-),1]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$

4. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{I},-]} [[\mathcal{I},\mathcal{A}_{x',y}],[\mathcal{I},\mathcal{A}_{x',z}]] \xrightarrow{[[1,ev_\star]} [[\mathcal{I},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \xrightarrow{[[f,1],1]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \cdots$
$\cdots \xrightarrow{[\mathcal{A}(x',-),1]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$

5. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[ev_\star,1]} [[\mathcal{I},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \xrightarrow{[[f,1],1]} [[\mathcal{A}_{x',x},\mathcal{A}_{x',y}],\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}(x',-),1]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$

6. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x',-)} [\mathcal{A}_{x',y},\mathcal{A}_{x',z}] \xrightarrow{[\mathcal{A}(f,1),1]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}].$

In the previous derivation arrows 2. and 3. are equal according to Lemma [Sch08]-9.11and arrows 4. and 5. are equal according to Lemma [Sch08]-11.3.

The codomain of $c^1{}_{f,y,z}$ rewrites

$$\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y},\mathcal{A}_{x,z}] \xrightarrow{[1,\mathcal{A}(x',-)]} [\mathcal{A}_{x,y},[\mathcal{A}_{x',x},\mathcal{A}_{x',z}]] \xrightarrow{[1,[f,1]]} [\mathcal{A}_{x,y},[\mathcal{I},\mathcal{A}_{x',z}]] \xrightarrow{[1,ev_\star]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$$
$$\mathcal{A}_{x,y} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y},\mathcal{A}_{x,z}] \xrightarrow{[1,\mathcal{A}(f,1)]} [\mathcal{A}_{x,y},\mathcal{A}_{x',z}]$$

Remark 7.24 *For any arrows $f : \mathcal{A} \to \mathcal{B}$ and $\tilde{f} : \mathcal{I} \to [\mathcal{A},\mathcal{B}]$ such that $ev_\star(\tilde{f}) = f$ and any object D in SPC, all the diagrams in the pasting below commute*

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}]} \arrow[r, "{[\mathcal{D},-]}"] \arrow[d, "{[f,\mathcal{B}]}"'] & {[[\mathcal{D},\mathcal{A}],[\mathcal{D},\mathcal{B}]]} \arrow[d, "{[\tilde{f},1]}"] \arrow[dl, "ev_f"'] \\
{[\mathcal{D},\mathcal{B}]} & {[\mathcal{I},[\mathcal{D},\mathcal{B}]].} \arrow[l, "ev_\star"]
\end{tikzcd}$$

The top left one commutes according to Corollary [Sch08]-11.6and the bottom right one does according to the 2-naturality of the collection of arrows q.

7.25 *Definition of the 2-cell* ${c^2}_{x,g:y'\to y,z}$.

Observe that the image by $ev_\star : SPC(\mathcal{I}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,y}]) \to SPC(\mathcal{A}_{x,y'}, \mathcal{A}_{x,y})$ of

$$\mathcal{I} \xrightarrow{\tilde{g}} \mathcal{A}_{y',y} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,y}]$$

is equal to $\mathcal{A}(1,g) : \mathcal{A}_{x,y'} \to \mathcal{A}_{x,y}$. Therefore according to Remark 7.24 the domain of this 2-cell which is

$$\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}] \xrightarrow{[\mathcal{A}_{x,y'},-]} [[\mathcal{A}_{x,y'}, \mathcal{A}_{x,y}], [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \xrightarrow{[\mathcal{A}(x,-),1]} [\mathcal{A}_{y,y}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \xrightarrow{[g,1]} [\mathcal{I}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \ldots$$
$$\ldots \xrightarrow{ev_\star} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$$

is equal to

$$\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}] \xrightarrow{[\mathcal{A}(1,g),1]} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$$

The codomain of the 2-cell ${c^2}_{x,g:y'\to y,z}$ rewrites successively

1. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(y'-)} [\mathcal{A}_{y',y}, \mathcal{A}_{y',z}] \xrightarrow{[1,\mathcal{A}(x,-)]} [\mathcal{A}_{y',y}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \xrightarrow{[g,1]} [\mathcal{I}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \xrightarrow{ev_\star} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$
2. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(y'-)} [\mathcal{A}_{y',y}, \mathcal{A}_{y',z}] \xrightarrow{[g,1]} [\mathcal{I}, \mathcal{A}_{y',z}] \xrightarrow{[1,\mathcal{A}(x,-)]} [\mathcal{I}, [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]] \xrightarrow{ev_\star} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$
3. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(y'-)} [\mathcal{A}_{y',y}, \mathcal{A}_{y',z}] \xrightarrow{[g,1]} [\mathcal{I}, \mathcal{A}_{y',z}] \xrightarrow{ev_\star} \mathcal{A}_{y',z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$
4. $\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(g,1)} \mathcal{A}_{y',z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y'}, \mathcal{A}_{x,z}]$

where in the above derivation arrows 2. and 3. are equal due to Lemma [Sch08]-11.2.

7.26 *Definition of the 2-cell* ${c^3}_{x,y,h:z\to z'}$.

The 2-cell

$$\mathcal{I} \xrightarrow{h} \mathcal{A}_{z,z'} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,z}, \mathcal{A}_{x,z'}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,z'}]] \xrightarrow{[\mathcal{A}(x,-),1]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,z'}]]$$

has image by $ev_\star$

$$\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}] \xrightarrow{[1,\mathcal{A}(1,h)]} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z'}]$$

which is the domain of ${c^3}_{x,y,h:z\to z'}$.
The 2-cell

$$\mathcal{I} \xrightarrow{h} \mathcal{A}_{z,z'} \xrightarrow{\mathcal{A}(y,-)} [\mathcal{A}_{y,z}, \mathcal{A}_{y,z'}] \xrightarrow{[1,\mathcal{A}(x,-)]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,z'}]]$$

has image by $ev_\star$

$$\mathcal{A}_{y,z} \xrightarrow{\mathcal{A}(1,h)} \mathcal{A}_{y,z'} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z'}]$$

which is the codomain of ${c^3}_{x,y,h:z\to z'}$.

7.27 *Proof of the equivalence of Axioms 4.4/3.14.*

The first of the 2-cell of Axiom 4.4 decomposes as the product $\Xi_2 \circ \Xi_1$ where Ξ_1 is

$$((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xrightarrow{A'} (\mathcal{A}_{t,u}\mathcal{A}_{z,t})(\mathcal{A}_{y,z}\mathcal{A}_{x,y}) \xrightarrow{1\otimes c_{x,y,z}} (\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{x,z} \xRightarrow{\alpha_{x,z,t,u}} \mathcal{A}_{x,u}$$

and Ξ_2 is

$$((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xrightarrow{(c_{z,t,u}\otimes 1)\otimes 1} (\mathcal{A}_{z,u}\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xRightarrow{\alpha_{x,y,z,u}} \mathcal{A}_{x,u}$$

The 2-cell Ξ_1 has a strict domain which image by Rn is strict since the arrows A', $Rn(A')$ and c are strict. According to Lemma [Sch08]-19.6, the 2-cell Ξ_1 has image by $Rn \circ Rn$

$$\mathcal{A}_{t,u} \otimes \mathcal{A}_{z,t} \xRightarrow{Rn(\alpha_{x,z,t,u})} [\mathcal{A}_{x,z}, \mathcal{A}_{x,u}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\mathcal{A}(x,-),1]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]].$$

This 2-cell has again strict domain and its image by Rn is the 2-cell

$$\mathcal{A}_{t,u} \xRightarrow{\alpha'} [\mathcal{A}_{z,t}, [\mathcal{A}_{x,z}, \mathcal{A}_{x,u}]] \xrightarrow{[1,[\mathcal{A}_{x,y},-]]} [\mathcal{A}_{z,t}, [[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]] \xrightarrow{[1,[\mathcal{A}(x,-),1]]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]].$$

The image by Rn of Ξ_2 is

$$(\mathcal{A}_{t,u} \otimes \mathcal{A}_{z,t}) \otimes \mathcal{A}_{y,z} \xrightarrow{c_{z,t,u}\otimes 1} \mathcal{A}_{z,u} \otimes \mathcal{A}_{y,z} \xRightarrow{Rn(\alpha_{x,y,z,u})} [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]$$

which has image by Rn the 2-cell

$$\mathcal{A}_{t,u} \otimes \mathcal{A}_{z,t} \xrightarrow{c_{z,t,u}} \mathcal{A}_{z,u} \xRightarrow{\alpha'_{x,y,z,u}} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]$$

which has image by Rn the 2-cell

$$\mathcal{A}_{t,u} \xrightarrow{\mathcal{A}(z,-)} [\mathcal{A}_{z,t}, \mathcal{A}_{z,u}] \xRightarrow{[1,\alpha'_{x,y,z,u}]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]].$$

The second 2-cell from Axiom 4.4 decomposes as $\Xi_5 \circ \Xi_4 \circ \Xi_3$ where Ξ_3 is

$$((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xrightarrow{A'\otimes 1} (\mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z}))\mathcal{A}_{x,y} \xrightarrow{A'} \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z}))\mathcal{A}_{x,y}) \xRightarrow{1\otimes\alpha_{x,y,z,t}} \mathcal{A}_{t,u}\mathcal{A}_{x,t} \xrightarrow{c} \mathcal{A}_{x,u}$$

Ξ_4 is

$$((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xrightarrow{A'\otimes 1} (\mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z}))\mathcal{A}_{x,y} \xrightarrow{(1\otimes c_{y,z,t})\otimes 1} (\mathcal{A}_{t,u}\mathcal{A}_{y,t})\mathcal{A}_{x,y} \xRightarrow{\alpha_{x,y,t,u}} \mathcal{A}_{x,u}$$

and Ξ_5 is

$$((\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z})\mathcal{A}_{x,y} \xRightarrow{\alpha_{y,z,t,u}\otimes 1} \mathcal{A}_{y,u}\mathcal{A}_{x,y} \xrightarrow{c_{x,y,u}} \mathcal{A}_{x,u}$$

The image by $Rn \circ Rn \circ Rn$ of the 2-cell Ξ_3 is

$$\mathcal{A}_{t,u} \xrightarrow{\eta} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z})] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z})] \xrightarrow{[1,[1,Rn(A')]]} \cdots$$

$$\cdots [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y})]]] \xRightarrow{[1,[1,[1,1\otimes\alpha_{x,y,z,t}]]]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}]]] \xrightarrow{[1,[1,[1,c_{x,t,u}]]]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$$

which rewrites

1. $\mathcal{A}_{t,u} \xrightarrow{\eta} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z})] \xrightarrow{[1,Rn(A')]} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y})]] \xRightarrow{[1,[1,1\otimes\alpha_{x,y,z,t}]]} \cdots$

   $\cdots [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}]] \xrightarrow{[1,[1,c_{x,t,u}]]} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$
2. $\mathcal{A}_{t,u} \xrightarrow{\eta} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y})] \xrightarrow{Rn} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y})]] \xRightarrow{[1,[1,1\otimes\alpha_{x,y,z,t}]]} \cdots$

   $\cdots [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}]] \xrightarrow{[1,[1,c_{x,t,u}]]} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]$

3. $\mathcal{A}_{t,u} \xrightarrow{\eta} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{t,u}((\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y})] \xRightarrow{[1,1\otimes\alpha_{x,y,z,t}]} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}] \xrightarrow{[1,c_{x,t,u}]} \cdots$

$\cdots [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{x,u}] \xrightarrow{Rn} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

4. $\mathcal{A}_{t,u} \xrightarrow{\eta} [\mathcal{A}_{x,t}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}] \xRightarrow{[\alpha_{x,y,z,t},1]} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{t,u}\mathcal{A}_{x,t}] \xrightarrow{[1,c_{x,t,u}]} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{x,u}] \xrightarrow{Rn} \cdots$

$\cdots [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

5. $\mathcal{A}_{t,u} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,t}, \mathcal{A}_{x,u}] \xrightarrow{[\alpha_{x,y,z,t},1]} [(\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}, \mathcal{A}_{x,u}] \xRightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, (\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\eta,1]} \cdots$

$\cdots [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

6. $\mathcal{A}_{t,u} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,t}, \mathcal{A}_{x,u}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,t}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xRightarrow{[[1,\alpha_{x,y,z,t}],1]} [[\mathcal{A}_{x,y}, (\mathcal{A}_{z,t}\mathcal{A}_{y,z})\mathcal{A}_{x,y}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\eta,1]} \cdots$

$\cdots [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{Rn} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

7. $\mathcal{A}_{t,u} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,t}, \mathcal{A}_{x,u}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,t}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xRightarrow{[Rn(\alpha_{x,y,z,t}),1]} [\mathcal{A}_{z,t}\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\mathcal{A}_{y,z},-]} \cdots$

$\cdots [[\mathcal{A}_{y,z}, \mathcal{A}_{z,t}\mathcal{A}_{y,z}], [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]] \xrightarrow{[\eta,1]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

8. $\mathcal{A}_{t,u} \xrightarrow{\mathcal{A}(x,-)} [\mathcal{A}_{x,t}, \mathcal{A}_{x,u}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,t}], [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\mathcal{A}_{y,z},-]} [[\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,t}]], [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]] \cdots$

$\cdots \xRightarrow{[\alpha'_{x,y,z,t},1]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$

The image by Rn of the 2-cell Ξ_4 is

$$(\mathcal{A}_{t,u}\mathcal{A}_{z,t})\mathcal{A}_{y,z} \xrightarrow{A'} \mathcal{A}_{t,u}(\mathcal{A}_{z,t}\mathcal{A}_{y,z}) \xrightarrow{1\otimes c_{y,z,t}} \mathcal{A}_{t,u}\mathcal{A}_{y,t} \xRightarrow{Rn(\alpha_{x,y,t,u})} [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]$$

which according to Lemma [Sch08]-19.6as image by $Rn \circ Rn$ the 2-cell

$$\mathcal{A}_{t,u} \xRightarrow{\alpha'_{x,y,t,u}} [\mathcal{A}_{y,t}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]] \xrightarrow{[\mathcal{A}_{y,z},-]} [[\mathcal{A}_{y,z}, \mathcal{A}_{y,t}], [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, A_{x,u}]]] \xrightarrow{[\mathcal{A}(y,-),1]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]].$$

The image by $Rn \circ Rn \circ Rn$ of the 2-cell Ξ_5 is

$$\mathcal{A}_{t,u} \xRightarrow{\alpha'_{y,z,t,u}} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, \mathcal{A}_{y,u}]] \xrightarrow{[1,[1,\mathcal{A}(x,-)]]} [\mathcal{A}_{z,t}, [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{A}_{x,u}]]]$$

∎

7.28 *Proof of the equivalence of Axioms 4.5 and 3.15.*

PROOF:It is easy to check that the image by Rn of the 2-cell Ξ_1 of Axiom 4.5 is $A(x,-)_{y,z} * \rho'_{y,z}$ the first 2-cell of the Axiom 3.15.

The image by Rn of the 2-cell Ξ_2 is

$$\begin{tikzcd}
\mathcal{A}_{y,z} \arrow[r, "\mathcal{A}(x,-)"] & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]} \arrow[rr, "id"] \arrow[d, "{[ev_\star,1]}"'] \arrow[drr, phantom, "\Downarrow {[\lambda',1]}"] & & {[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]} \\
& {[[\mathcal{I}, \mathcal{A}_{x,y}], \mathcal{A}_{x,z}]} \arrow[rr, "{[[u_y,1],1]}"'] & & {[[\mathcal{A}_{y,y}, \mathcal{A}_{x,y}], \mathcal{A}_{x,z}]} \arrow[u, "{[\mathcal{A}(-,y),1]}"']
\end{tikzcd}$$

The 2-cell Ξ_3, namely

$$\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y} \xrightarrow{R'\otimes 1} (\mathcal{A}_{y,z} \otimes \mathcal{I}) \otimes \mathcal{A}_{x,y} \xrightarrow{(1\otimes u_y)\otimes 1} \mathcal{A}_{y,z} \otimes \mathcal{A}_{y,y} \xRightarrow{\alpha_{x,y,y,z}} \mathcal{A}_{x,z}$$

has image by Rn the 2-cell

$$\mathcal{A}_{y,z} \xrightarrow{R'} \mathcal{A}_{y,z} \otimes \mathcal{I} \xrightarrow{1\otimes u_y} \mathcal{A}_{y,z} \otimes \mathcal{A}_{y,y} \xrightarrow{Rn(\alpha_{x,y,y,z})} [\mathcal{A}_{x,y}, \mathcal{A}_{x,z}]$$

which is according to Lemma 7.9

$$\mathcal{A}_{y,z} \xRightarrow{\alpha'_{x,y,y,z}} [\mathcal{A}_{y,y},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]] \xrightarrow{[u_y,1]} [\mathcal{I},[\mathcal{A}_{x,y},\mathcal{A}_{x,z}]] \xrightarrow{[ev_\star,1]} [\mathcal{A}_{x,y},\mathcal{A}_{x,z}]$$

$\blacksquare$

7.29 *Equivalence of Axioms 4.7 and 3.18.*

First let us remark that the two 2-cells of Axiom 3.18 have the same domain. This results from the following sequence of equal arrows.

1. $[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{B}_{Fx,Fy},-]} [[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[\mathcal{B}(Fx,-),1]} [\mathcal{B}_{Fy,Fz},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[F_{y,z},1]} \cdots$

   $\cdots [\mathcal{A}_{y,z},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[1,[F_{x,y},1]]} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]$
2. $[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{B}_{Fx,Fy},-]} [[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[\mathcal{B}(Fx,-),1]} [\mathcal{B}_{Fy,Fz},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[1,[F_{x,y},1]]} \cdots$

   $\cdots [\mathcal{B}_{Fy,Fz},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[F_{y,z},1]} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]$
3. $[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{B}_{Fx,Fy},-]} [[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[1,[F_{x,y},1]]} [[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[\mathcal{B}(Fx,-),1]} \cdots$

   $\cdots [\mathcal{B}_{Fy,Fz},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[F_{y,z},1]} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]]$
4. $[\mathcal{B}_{Fx,Fz},\mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Fz}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[[F_{x,y},1],1]} [[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Fz}],[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[\mathcal{B}(Fx,-),1]} \cdots$

   $\cdots [\mathcal{B}_{Fy,Fz},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[F_{y,z},1]} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]].$

In the above derivation arrows 3. and 4. are equal according to Lemma [Sch08]-9.11.

The 2-cell

$$(\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z})\otimes\mathcal{A}_{x,y} \xrightarrow{(F_{z,t}\otimes F_{y,z})\otimes 1} (\mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fy,Fz})\otimes\mathcal{A}_{x,y} \xrightarrow{1\otimes F_{x,y}} (\mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fy,Fz})\otimes\mathcal{B}_{Fx,Fy} \xRightarrow{\alpha_{Fx,Fy,Fz,Ft}} \mathcal{B}_{Fx,Ft}$$

has a strict domain and image by Rn

$$\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z} \xrightarrow{F_{z,t}\otimes F_{y,z}} \mathcal{B}_{Fz,Ft}\otimes\mathcal{B}_{Fy,Fz} \xrightarrow{Rn(\alpha'_{Fx,Fy,Fz,Ft})} [\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}] \xRightarrow{[F_{x,y},1]} [\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]$$

which has a strict domain and image by Rn

$$\mathcal{A}_{z,t} \xrightarrow{F_{z,t}} \mathcal{B}_{Fz,Ft} \xRightarrow{\alpha'} [\mathcal{B}_{Fy,Fz},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[F_{y,z},1]} [\mathcal{A}_{y,z},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[1,[F_{x,y},1]]} [\mathcal{A}_{y,z},[\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]].$$

The 2-cell

$$(\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z})\otimes\mathcal{A}_{x,y} \xRightarrow{F^2_{y,z,t}\otimes 1} \mathcal{B}_{Fy,Ft}\otimes\mathcal{A}_{x,y} \xrightarrow{1\otimes F_{x,y}} \mathcal{B}_{Fy,Ft}\otimes\mathcal{B}_{Fx,Fy} \xrightarrow{c} \mathcal{B}_{Fx,Ft}$$

has image by Rn that rewrites

$$\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z} \xRightarrow{F^2} \mathcal{B}_{Fy,Ft} \xrightarrow{Rn(c\otimes(1\otimes F_{x,y}))} [\mathcal{A}_{x,y},\mathcal{B}_{Fx,Ft}]$$

$$\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z} \xRightarrow{F^2} \mathcal{B}_{Fy,Ft} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}] \xrightarrow{[F_{x,y},1]} [\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]$$

The image by Rn of this last arrow is

$$\mathcal{A}_{z,t} \xRightarrow{F'^2} [\mathcal{A}_{y,z},\mathcal{B}_{Fy,Ft}] \xrightarrow{[1,\mathcal{B}(Fx,-)]} [\mathcal{A}_{y,z},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]] \xrightarrow{[1,[F_{x,y},1]]} [\mathcal{A}_{y,z},[\mathcal{B}_{Fx,Fy},\mathcal{B}_{Fx,Ft}]].$$

The 2-cell

$$(\mathcal{A}_{z,t}\otimes\mathcal{A}_{y,z})\otimes\mathcal{A}_{x,y} \xrightarrow{c\otimes 1} \mathcal{A}_{y,t}\otimes\mathcal{A}_{x,y} \xRightarrow{F^2_{x,y,t}} \mathcal{B}_{Fx,Ft}$$

has image by Rn the 2-cell

$$\mathcal{A}_{z,t} \otimes \mathcal{A}_{y,z} \xrightarrow{c} \mathcal{A}_{y,t} \xRightarrow{F'^2} [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Ft}]$$

which has image by Rn

$$\mathcal{A}_{z,t} \xrightarrow{\mathcal{A}(y,-)} [\mathcal{A}_{y,z}, \mathcal{A}_{y,t}] \xRightarrow{[1,F'^2_{x,y,t}]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Ft}]].$$

The 2-cell

$$\mathcal{A}_{z,t} \otimes \mathcal{B}_{Fx,Fz} \xrightarrow{F_{z,t} \otimes 1} \mathcal{B}_{Fz,Ft} \otimes \mathcal{B}_{Fx,Fz} \xrightarrow{c_{Fx,Fz,Ft}} \mathcal{B}_{Fx,Ft}$$

has image by Rn the 2-cell

$$\mathcal{A}_{z,t} \xrightarrow{F_{z,t}} \mathcal{B}_{Fz,Ft} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fz}, \mathcal{B}_{Fx,Ft}]$$

therefore according to Lemma 7.21 the 2-cell

$$(\mathcal{A}_{z,t} \otimes \mathcal{A}_{y,z}) \otimes \mathcal{A}_{x,y} \xrightarrow{A'} \mathcal{A}_{z,t} \otimes (\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y}) \xRightarrow{1 \otimes F^2_{x,y,z}} \mathcal{A}_{z,t} \otimes \mathcal{B}_{Fx,Fz} \xrightarrow{F_{z,t} \otimes 1} \mathcal{B}_{Fz,Ft} \otimes \mathcal{B}_{Fx,Fz} \xrightarrow{c_{Fx,Fz,Ft}} \mathcal{B}_{Fx,Ft}$$

has image by $Rn \circ Rn$ the 2-cell

$$\mathcal{A}_{z,t} \xrightarrow{F_{z,t}} \mathcal{B}_{Fz,Ft} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fz}, \mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Fz}], [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Ft}]] \xRightarrow{[F'^2_{x,y,z},1]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Ft}]].$$

According to Lemma 7.21 the 2-cell

$$(\mathcal{A}_{z,t} \otimes \mathcal{A}_{y,z}) \otimes \mathcal{A}_{x,y} \xrightarrow{A'} \mathcal{A}_{z,t} \otimes (\mathcal{A}_{y,z} \otimes \mathcal{A}_{x,y}) \xrightarrow{1 \otimes c_{x,y,z}} \mathcal{A}_{z,t} \otimes \mathcal{A}_{x,z} \xRightarrow{F^2_{x,z,t}} \mathcal{B}_{Fx,Ft}$$

has image by $Rn \circ Rn$ the 2-cell

$$\mathcal{A}_{z,t} \xRightarrow{F'^2_{x,z,t}} [\mathcal{A}_{x,z}, \mathcal{B}_{Fx,Ft}] \xrightarrow{[\mathcal{A}_{x,y},-]} [[\mathcal{A}_{x,y}, \mathcal{A}_{x,z}], [\mathcal{A}_{x,y} \mathcal{B}_{Fx,Ft}]] \xrightarrow{[\mathcal{A}(x,-),1]} [\mathcal{A}_{y,z}, [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Ft}]].$$

∎

7.30 *Equivalence of Axioms 4.8 and 3.19.*

PROOF:According to Lemma 7.9, the arrow

$$\mathcal{A}_{x,y} \xrightarrow{R'} \mathcal{A}_{x,y} \otimes \mathcal{I} \xrightarrow{1 \otimes u_x} \mathcal{A}_{x,y} \otimes \mathcal{A}_{x,x} \xRightarrow{F^2_{x,x,y}} \mathcal{A}_{x,y}$$

is equal to

$$\mathcal{A}_{x,y} \xRightarrow{F'^2_{x,x,y}} [\mathcal{A}_{x,x}, \mathcal{B}_{Fx,Fy}] \xrightarrow{[u_x,1]} [\mathcal{I}, \mathcal{B}_{Fx,Fy}] \xrightarrow{ev_\star} \mathcal{B}_{Fx,Fy}.$$

Since the image by Rn of

$$\mathcal{A}_{x,y} \otimes \mathcal{B}_{Fx,Fx} \xrightarrow{F_{x,y} \otimes 1} \mathcal{B}_{Fx,Fy} \otimes \mathcal{B}_{Fx,Fx} \xrightarrow{c} \mathcal{B}_{Fx,Fy}$$

is

$$\mathcal{A}_{x,y} \xrightarrow{F_{x,y}} \mathcal{B}_{Fx,Fy} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fx}, \mathcal{B}_{Fx,Fy}],$$

the arrow

$$\mathcal{A}_{x,y} \xrightarrow{R'} \mathcal{A}_{x,y} \otimes \mathcal{I} \xRightarrow{1\otimes F^0_x} \mathcal{A}_{x,y} \otimes \mathcal{B}_{Fx,Fx} \xrightarrow{F_{x,y}\otimes 1} \mathcal{B}_{Fx,Fy} \otimes \mathcal{B}_{Fx,Fx} \xrightarrow{c} \mathcal{B}_{Fx,Fy}$$

is equal to

$$\mathcal{A}_{x,y} \xrightarrow{F_{x,y}} \mathcal{B}_{Fx,Fy} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fx}, \mathcal{B}_{Fx,Fy}] \xRightarrow{[F^0_x,1]} [\mathcal{I}, \mathcal{B}_{Fx,Fy}] \xrightarrow{ev_\star} \mathcal{B}_{Fx,Fy}$$

according to Lemma 7.9. ∎

7.31 *Equivalence of Axioms 4.9 and 3.20.*

PROOF:The arrow

$$\mathcal{B}_{Fy,Fy} \otimes \mathcal{A}_{x,y} \xrightarrow{1\otimes F_{x,y}} \mathcal{B}_{Fy,Fy} \otimes \mathcal{B}_{Fx,Fy} \xrightarrow{c} \mathcal{B}_{Fx,Fy}$$

has image by Rn

$$\mathcal{B}_{Fy,Fy} \xrightarrow{\mathcal{B}(Fx,-)} [\mathcal{B}_{Fx,Fy}, \mathcal{B}_{Fx,Fy}] \xrightarrow{[F_{x,y},1]} [\mathcal{A}_{x,y}, \mathcal{B}_{Fx,Fy}]$$

which has dual

$$\mathcal{A}_{x,y} \xrightarrow{F_{x,y}} \mathcal{B}_{Fx,Fy} \xrightarrow{\mathcal{B}(-,Fy)} [\mathcal{B}_{Fy,Fy}, \mathcal{B}_{Fx,Fy}].$$

Therefore according to Lemma 7.8, the 2-cell

$$\mathcal{A}_{x,y} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{A}_{x,y} \xRightarrow{F^0_y\otimes 1} \mathcal{B}_{Fy,Fy} \otimes \mathcal{A}_{x,y} \xrightarrow{1\otimes F_{x,y}} \mathcal{B}_{Fy,Fy} \otimes \mathcal{B}_{Fx,Fy} \xrightarrow{c} \mathcal{B}_{Fx,Fy}$$

is equal to

$$\mathcal{A}_{x,y} \xrightarrow{F_{x,y}} \mathcal{B}_{Fx,Fy} \xrightarrow{\mathcal{B}(-,Fy)} [\mathcal{B}_{Fy,Fy}, \mathcal{B}_{Fx,Fy}] \xRightarrow{[F^0_y,1]} [\mathcal{I}, \mathcal{B}_{Fx,Fy}] \xrightarrow{ev_\star} \mathcal{B}_{Fx,Fy}.$$

According to Lemma 7.8 the 2-cell

$$\mathcal{A}_{x,y} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{A}_{x,y} \xrightarrow{u_y\otimes 1} \mathcal{A}_{y,y} \otimes \mathcal{A}_{x,y} \xrightarrow{F^2_{x,y,y}} \mathcal{A}_{x,y}$$

is equal to

$$\mathcal{A}_{x,y} \xrightarrow{({F'}^2_{x,y,y})^*} [\mathcal{A}_{y,y}, \mathcal{A}_{x,y}] \xrightarrow{[u_y,1]} [\mathcal{I}, \mathcal{A}_{x,y}] \xrightarrow{ev_\star} \mathcal{A}_{x,y}$$

∎

Section 5.

We will need the following characterization of bilinear natural transformations.

Remark 7.32 *For any symmetric monoidal functors $F, G : \mathcal{A} \to [\mathcal{B}, \mathcal{C}]$ with respective underlying functors $F', G' : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. Any 2-cell $\sigma : F \to G : \mathcal{A} \to [\mathcal{B}, \mathcal{C}]$ in SPC corresponds to a collection of arrows $\sigma_{a,b} : F'(a,b) \to G'(a,b)$ in $\mathcal{C}$, natural in a and b and that satisfies the following two conditions:*
- For all objects a, a', b and b' of $\mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F'(a,b) + F'(a,b') & \xrightarrow{{F^*}^2_{b,b'\,a}} & F'(a, b+b') \\ \Big\downarrow{\scriptstyle \sigma_{a,b}+\sigma_{a,b'}} & & \Big\downarrow{\scriptstyle \sigma_{a,b+b'}} \\ F'(a',b) + F'(a',b') & \xrightarrow{{F^*}^2_{b,b'\,a}{}'} & F'(a', b+b'); \end{array}$$

- For all objects a, a', b and b' of $\mathcal{A}$ the following diagram commutes

$$
\begin{tikzcd}
F'(a,b)+F'(a',b) \arrow[r, "F^2_{a,a'\,b}"] \arrow[d, "\sigma_{a,b}+\sigma_{a',b}"'] & F'(a+a',b) \arrow[d, "\sigma_{a+a',b}"'] \\
F'(a,b')+F'(a',b') \arrow[r, "F^2_{a,a'\,b'}"'] & F'(a+a',b')
\end{tikzcd}
$$

7.33 *Proof that 2-rings in the sense of 5.1 are exactly one point SPC-categories.*

PROOF:Let us start with a 2-ring $\mathcal{A}$ as defined in 5.1. Its corresponds to a one-point SPC-category as follows. Since monoidal functors $\mathcal{I} \to \mathcal{A}$ are in one-to-one correspondence with objects of $\mathcal{A}$, the object 1 correspond to a strict arrow $u : \mathcal{I} \to \mathcal{A}$ in SPC. That the multiplication ".", which is already a functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, defines an arrow $\varphi' : \mathcal{A} \to [\mathcal{A}, \mathcal{A}]$ in SPC corresponds to the existence of the natural arrows $\underline{a}_{b,b'}$ and $\overline{b}_{a,a'}$ and the commutation of Diagrams 5.2, 5.3, 5.4, 5.5 and 5.6. Precisely for any object a of $\mathcal{A}$, the multiplication on the left by a, defines a functor $a.- : \mathcal{A} \to \mathcal{A}$. That this one is monoidal corresponds to the existence of the arrows $\underline{a}_{b,b'}$ natural in b and b' and such that Diagrams 5.2 commute for all b, b' and b''. That the monoidal $a.-$ is symmetric corresponds to the commutation of 5.4 for all b and b'. That for any arrow $f : a \to a'$, the natural transformation $f.- : a.- \to a'.- : \mathcal{A} \to \mathcal{A}$ is monoidal for the structures described previously on $a.-$ and $a'.-$ corresponds to the naturality in the argument a of the maps $\underline{a}_{b,b'}$. The assignments $a \mapsto a.-$ and $(f : a \to a') \mapsto (f.- : a.- \to a'.-)$ define therefore a functor say $\varphi' : \mathcal{A} \to SPC(\mathcal{A}, \mathcal{A})$. The existence of a natural transformation $(a.-) + (a'.-) \to (a + a').-$ corresponds to the existence of arrows $\overline{b}_{a,a'}$ natural in b. This transformation is monoidal since Diagrams 5.6 commute. Eventually that the collection of these arrows for all a and a' defines a symmetric monoidal structure on the above functor φ' results from the the naturality of the collection in the arguments a and a' and the commutation Diagrams 5.3 and 5.5.

According to Remark 7.32, a 2-cell α' in SPC as in 3.11 corresponds to a natural collection of arrows $\tilde{\alpha}_{a,b,c} : a.(b.c) \to (a.b).c$ in $\mathcal{A}$ such that Diagrams 5.7, 5.8 and 5.9 commute. A 2-cell $\rho' : 1 \to ev_\star \circ [u, 1] \circ \varphi' : \mathcal{A} \to \mathcal{A}$ in SPC corresponds to a natural collection of arrows $\tilde{\rho}_a : a.1 \to a$ such that Diagrams 5.10 commute. Similarly a 2-cell $\lambda' : 1 \to ev_\star \circ [u, 1] \circ \varphi'^* : \mathcal{A} \to \mathcal{A}$ amounts to a natural collection of arrows $\tilde{\lambda}_a : 1.a \to a$ such that Diagrams 5.11 commute.

Then the coherence conditions 3.14 and 3.15 for α', ρ' and λ' above are equivalent to the coherence conditions for the associativity and unit laws of the monoidal category $(\mathcal{A}, ., I, \tilde{\alpha}, \tilde{\rho}, \tilde{\lambda})$. ∎

7.34 *Proof that 2-ring morphisms in the sense of 5.14 are exactly SPC-functors.*

PROOF:According to the two observations below the result becomes clear after inspection of Axioms 3.18, 3.19 and 3.20.

Observe first that according to Remark 7.32 a 2-cell

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] \arrow[dd, "H"'] & {[\mathcal{A},\mathcal{A}]} \arrow[dr, "{[1,H]}"] & \\
 & & {[\mathcal{A},\mathcal{B}]} \\
\mathcal{B} \arrow[r, "\varphi'"'] \arrow[uur, Rightarrow] & {[\mathcal{B},\mathcal{B}]} \arrow[ur, "{[\varphi',1]}"'] &
\end{tikzcd}
$$

in SPC is the same thing than a collection of arrows $\Xi_{a,b} : Ha.Hb \to H(a+b)$ natural in a and b and satisfying the conditions that for any objects a, b, c of $\mathcal{A}$, the two diagrams in $\mathcal{B}$ below commute

7.35

$$\begin{tikzcd}
H(a).H(b)+H(a).H(c) \arrow[r, "\overline{H(a)}_{H(b),H(c)}"] \arrow[d, "\Xi_{a,b}+\Xi_{a,c}"'] & H(a).(H(b)+H(c)) \arrow[r, "H(a).H^2_{b,c}"] & H(a).H(b+c) \arrow[d, "\Xi_{a,b+c}"] \\
H(a.b)+H(a.c) \arrow[r, "H^2_{a.b,a.c}"'] & H(a.b+a.c) \arrow[r, "H(\underline{a}_{b,c})"'] & H(a.(b+c))
\end{tikzcd}$$

7.36

$$\begin{tikzcd}
H(a).H(c)+H(b).H(c) \arrow[r, "\overline{H(c)}_{H(a),H(b)}"] \arrow[d, "\Xi_{a,c}+\Xi_{b,c}"'] & (H(a)+H(b)).H(c) \arrow[r, "H^2_{a,b}.H(c)"] & H(a+b).H(c) \arrow[d, "\Xi_{a+b,c}"] \\
H(a.c)+H(b.c) \arrow[r, "H^2_{a.c,b.c}"'] & H(a.c+b.c) \arrow[r, "H(\overline{c}_{a,b})"'] & H((a+b).c)
\end{tikzcd}$$

where we write the φ' as products. Also according to Remark 7.10 in Appendix any 2-cell

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dd, "H"] \\
\mathcal{I} \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[r, Rightarrow] & \\
 & \mathcal{B}
\end{tikzcd}$$

in SPC is fully determined by its component at the generator $\star$ which is an arrow $1_{\mathcal{B}} \to H(1_{\mathcal{A}})$ if $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ denote the images $u(\star)$, and conversely every such arrow corresponds to a 2-cell as above in this way. ∎

7.37 *Proof of Proposition 5.17.*

PROOF:That there are identity 2-cells 3.11 amounts to the fact that the diagram in SPC

$$\begin{tikzcd}
 & {[\mathcal{C},\mathcal{D}]} \arrow[dl, "{[\mathcal{B},-]}"'] \arrow[dr, "{[\mathcal{A},-]}"] & \\
{[[\mathcal{B},\mathcal{C}],[\mathcal{B},\mathcal{D}]]} \arrow[d, "{[1,[\mathcal{A},-]]}"'] & & {[[\mathcal{A},\mathcal{C}],[\mathcal{A},\mathcal{D}]]} \arrow[d, "{[[\mathcal{A},\mathcal{B}],-]}"] \\
{[[\mathcal{B},\mathcal{C}],[[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{D}]]} & & {[[[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{C}]],[[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{D}]]} \arrow[ll, "{[[\mathcal{A},-],1]}"]
\end{tikzcd}$$

commutes for any objects $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. Such a diagram involves only strict arrows in SPC and

its underlying diagram in **Cat** is

$$\begin{tikzcd}
 & SPC(\mathcal{C},\mathcal{D}) \arrow[dl, "Post"'] \arrow[dr, "Post"] & \\
SPC([\mathcal{B},\mathcal{C}],[\mathcal{B},\mathcal{D}]) \arrow[d, "SPC(1,[\mathcal{A},-])"'] & & SPC([\mathcal{A},\mathcal{C}],[\mathcal{A},\mathcal{D}]) \arrow[d, "Post"] \\
SPC([\mathcal{B},\mathcal{C}],[[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{D}]]) & & SPC([[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{C}]],[[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{D}]]) \arrow[ll, "SPC([\mathcal{A},-],1)"]
\end{tikzcd}$$

which commutes according to Lemma [Sch08]-9.10.

One has identity 2-cells ρ' since for any objects $\mathcal{A}$ and $\mathcal{B}$ in SPC the composite

$$[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{B}]] \xrightarrow{[v,1]} [\mathcal{I},[\mathcal{A},\mathcal{B}]] \xrightarrow{ev_\star} [\mathcal{A},\mathcal{B}]$$

which is strict and has underlying functor that is an identity, hence is an identity in SPC.

One has identity 2-cells λ' since the composite

$$[\mathcal{A},\mathcal{B}] \xrightarrow{[-,\mathcal{B}]} [[\mathcal{B},\mathcal{B}],[\mathcal{A},\mathcal{B}]] \xrightarrow{[v,1]} [\mathcal{I},[\mathcal{A},\mathcal{B}]] \xrightarrow{ev_\star} [\mathcal{A},\mathcal{B}]$$

is the identity at $[\mathcal{A},\mathcal{B}]$ as shown below. The arrow $[\mathcal{A},\mathcal{B}] \xrightarrow{[-,\mathcal{B}]} [[\mathcal{B},\mathcal{B}],[\mathcal{A},\mathcal{B}]] \xrightarrow{[v,1]} [\mathcal{I},[\mathcal{A},\mathcal{B}]]$ is v^* since it has dual

$$\mathcal{I} \xrightarrow{v} [\mathcal{B},\mathcal{B}] \xrightarrow{[\mathcal{A},-]} [[\mathcal{A},\mathcal{B}],[\mathcal{A},\mathcal{B}]]$$

that is v according to Lemma [Sch08]-18.5. One concludes since $ev_\star \circ v^* = 1$. ∎

7.38 *Proof of Proposition 5.19*

PROOF:That one has an identity 2-cell 3.11 amounts to the commutation of the diagram

$$\begin{tikzcd}
 & \mathcal{I} \arrow[dl, "v"'] \arrow[dr, "v"] & \\
[\mathcal{I},\mathcal{I}] \arrow[d, "{[\mathcal{I},-]}"'] & & [\mathcal{I},\mathcal{I}] \arrow[d, "{[1,v]}"] \\
[[\mathcal{I},\mathcal{I}],[\mathcal{I},\mathcal{I}]] \arrow[rr, "{[v,1]}"'] & & [\mathcal{I},[\mathcal{I},\mathcal{I}]].
\end{tikzcd}$$

This diagram commutes since the arrow

$$\mathcal{I} \xrightarrow{v} [\mathcal{I},\mathcal{I}] \xrightarrow{[\mathcal{I},-]} [[\mathcal{I},\mathcal{I}],[\mathcal{I},\mathcal{I}]] \xrightarrow{[v,1]} [\mathcal{I},[\mathcal{I},\mathcal{I}]]$$

is

$$\mathcal{I} \xrightarrow{v} [[\mathcal{I},\mathcal{I}],[\mathcal{I},\mathcal{I}]] \xrightarrow{[v,1]} [\mathcal{I},[\mathcal{I},\mathcal{I}]]$$

according to Lemma [Sch08]-18.5, which is equal to

$$\mathcal{I} \xrightarrow{v} [\mathcal{I},\mathcal{I}] \xrightarrow{[1,v]} [\mathcal{I},[\mathcal{I},\mathcal{I}]].$$

according Lemma [Sch08]-18.8.

One has an identity 2-cell ρ' since the composite $\mathcal{I} \xrightarrow{v} [\mathcal{I}, \mathcal{I}] \xrightarrow{ev_\star} \mathcal{I}$ is the identity.

One has also an identity 2-cell λ' since the composite

$$\mathcal{I} \xrightarrow{v^*} [\mathcal{I}, \mathcal{I}] \xrightarrow{ev} \mathcal{I}$$

is an identity according to Lemma [Sch08]-18.4. ∎

7.39 *Proof of Lemma 5.18.*

PROOF:According to Lemma [Sch08]-11.4, the diagram in SPC

$$\begin{tikzcd}
{[\mathcal{I}, [\mathcal{A}, \mathcal{B}]]} \arrow[r, "ev_\star"] \arrow[d, "{[-,[\mathcal{A},\mathcal{C}]]}"'] & {[\mathcal{A}, \mathcal{B}]} \arrow[d, "q"] \\
{[[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]], [\mathcal{I}, [\mathcal{A}, \mathcal{C}]]} \arrow[r, "{[1,ev_\star]}"'] & {[[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]], [\mathcal{A}, \mathcal{C}]]}
\end{tikzcd}$$

commutes, therefore also does the diagram

$$\begin{tikzcd}
{[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]} \arrow[r, "{[\tilde{F},1]}"] \arrow[dr, "ev_\star"'] & {[\mathcal{I}'[\mathcal{A}, \mathcal{C}]]} \arrow[d, "ev_\star"] \\
 & {[\mathcal{A}, \mathcal{C}].}
\end{tikzcd}$$

From this and according to Corollary [Sch08]-11.6all diagrams in the pasting below

$$\begin{tikzcd}
{[\mathcal{B}, \mathcal{C}]} \arrow[r, "{[F,\mathcal{C}]}"] \arrow[d, "{[\mathcal{A},-]}"'] & {[\mathcal{A}, \mathcal{C}]} \\
{[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]} \arrow[r, "{[\tilde{F},1]}"'] \arrow[ur, "ev_F"] & {[\mathcal{I}, [\mathcal{A}, \mathcal{C}]]} \arrow[u, "ev_\star"']
\end{tikzcd}$$

commute and according to Lemma 7.9 the commutation of the external diagram above is equivalent to the commutation of the first diagram of the Lemma.
In the pasting

$$\begin{tikzcd}
{[\mathcal{C}, \mathcal{A}]} \arrow[r, "{[\mathcal{C},F]}"] \arrow[d, "{[-,\mathcal{B}]}"'] & {[\mathcal{C}, \mathcal{B}]} \\
{[[\mathcal{A}, \mathcal{B}], [\mathcal{C}, \mathcal{B}]]} \arrow[r, "{[\tilde{F},1]}"'] \arrow[ur, "ev_F"] & {[\mathcal{I}, [\mathcal{C}, \mathcal{B}]]} \arrow[u, "ev_\star"']
\end{tikzcd}$$

the top-left diagram commutes according to Corollary [Sch08]-11.8and we have already seen that the bottom-left diagram commutes. The commutation of the external diagram above is equivalent to the commutation of the second diagram of the Lemma according to Lemma 7.8. ∎

Section 6.

7.40 *Definition of the 2-cell θ from Axiom 6.22.*

PROOF:The arrows

$$\mathcal{A} \otimes \mathcal{M} \xrightarrow{L' \otimes 1} (\mathcal{I} \otimes \mathcal{A}) \otimes \mathcal{M} \xrightarrow{A'} \mathcal{I} \otimes (\mathcal{A} \otimes \mathcal{M})$$

is strict and, as shown below, it has the same image by Rn as the arrow $L' : \mathcal{A} \otimes \mathcal{M} \to \mathcal{I} \otimes (\mathcal{A} \otimes \mathcal{M})$. The 2-cell θ corresponds then via the adjunction 2.5 to the identity 2-cell.
The image by Rn of $L'_{\mathcal{A}}$ is

$$\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\eta^*]} [\mathcal{M}, [\mathcal{I}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{[1,ev_\star]} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})].$$

The arrow $A' \circ L' \otimes 1$ above as image by Rn that rewrites

1. $\mathcal{A} \xrightarrow{L'} \mathcal{I}\mathcal{A} \xrightarrow{Rn(A')} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
2. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{I}, \mathcal{I}\mathcal{A}] \xrightarrow{ev_\star} \mathcal{I}\mathcal{A} \xrightarrow{Rn(A')} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
3. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{I}, \mathcal{I}\mathcal{A}] \xrightarrow{[1, Rn(A')]} [\mathcal{I}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
4. $\mathcal{A} \xrightarrow{Rn(Rn(A'))^*} [\mathcal{I}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
5. $\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[-,\mathcal{I}(\mathcal{A}\mathcal{M})]} [[\mathcal{A}\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})], [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{[\eta,1]} [\mathcal{I}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
6. $\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\eta^*]} [\mathcal{M}, [\mathcal{I}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{D} [\mathcal{I}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$
7. $\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\eta^*]} [\mathcal{M}, [\mathcal{I}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{[1,ev_\star]} [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]$

where in the above derivation the equalities between arrows hold for the following reasons:
- 4. and 5. by definition of A' since its image by $Rn \circ Rn$ is in this case
$\mathcal{I} \xrightarrow{\eta} [\mathcal{A}\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{A}\mathcal{M}], [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{[\eta,1]} [\mathcal{A}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]]$ that has dual
$\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[-,\mathcal{I}(\mathcal{A}\mathcal{M})]} [[\mathcal{A}\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})], [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]] \xrightarrow{[\eta,1]} [\mathcal{I}, [\mathcal{M}, \mathcal{I}(\mathcal{A}\mathcal{M})]]$;
- 5. and 6. by Lemma [Sch08]-10.8;
- 6. and 7. by Lemma [Sch08]-11.9. ∎

7.41 *Proof of 6.26.*

PROOF:The first of the 2-cells of Axiom 6.20 can be decomposed as the composite $\Xi_2 \circ \Xi_1$ where Ξ_1 is

$$((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \xrightarrow{A'} (\mathcal{A}\mathcal{A})(\mathcal{A}\mathcal{M}) \xrightarrow{1 \otimes \varphi} (\mathcal{A}\mathcal{A})\mathcal{M} \overset{\beta}{\Longrightarrow} \mathcal{M}$$

and Ξ_2 is

$$((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \xrightarrow{(c \otimes 1) \otimes 1} (\mathcal{A}\mathcal{A})\mathcal{M} \overset{\beta}{\Longrightarrow} \mathcal{M}.$$

The 2-cell of 6.26 decomposes as $\Xi_4 \circ \Xi_3$ where Ξ_3 is

$$(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \overset{\beta'' \otimes 1}{\Longrightarrow} [\mathcal{M}, \mathcal{M}] \otimes \mathcal{A} \xrightarrow{1 \otimes \varphi'} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

and Ξ_4 is

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{c \otimes 1} \mathcal{A}\mathcal{A} \overset{\beta''}{\Longrightarrow} [\mathcal{M}, \mathcal{M}].$$

The 2-cell Ξ_3 has a strict domain and an easy computation gives that its image by Rn is

$$\mathcal{A} \otimes \mathcal{A} \overset{\beta''}{\Longrightarrow} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]]] \xrightarrow{[\varphi',1]} [\mathcal{A}, [\mathcal{A}, [\mathcal{M}, \mathcal{M}]]].$$

According to Lemma 7.8 the 2-cell Ξ_1 has an image by Rn with a strict domain and has image by $Rn \circ Rn$ the 2-cell

$$\mathcal{A}\mathcal{A} \xRightarrow{Rn(\beta)} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xrightarrow{[\varphi',1]} [\mathcal{A}, [\mathcal{M}, \mathcal{M}]].$$

Therefore the 2-cell Ξ_1 has image by Rn the 2-cell Ξ_3.

An easy computation shows that the 2-cell Ξ_2 has image by Rn

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{c\otimes 1} \mathcal{A}\mathcal{A} \xRightarrow{Rn(\beta)} [\mathcal{M}, \mathcal{M}]$$

which is Ξ_4. ∎

7.42 *Proof of 6.27.*

PROOF:The second 2-cells of Axiom 6.20 can be decomposed as the composite $\Xi_3 \circ \Xi_2 \circ \Xi_1$ where Ξ_1 is

$$((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \xrightarrow{A'\otimes 1} (\mathcal{A}(\mathcal{A}\mathcal{A}))\mathcal{M} \xrightarrow{A'} \mathcal{A}((\mathcal{A}\mathcal{A})\mathcal{M}) \xRightarrow{1\otimes\beta} \mathcal{A}\mathcal{M} \xrightarrow{\varphi} \mathcal{M}$$

Ξ_2 is

$$((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \xrightarrow{A'\otimes 1} (\mathcal{A}(\mathcal{A}\mathcal{A}))\mathcal{M} \xrightarrow{(1\otimes c)\otimes 1} (\mathcal{A}\mathcal{A})\mathcal{M} \xRightarrow{\beta} \mathcal{M}$$

and Ξ_3 is

$$((\mathcal{A}\mathcal{A})\mathcal{A})\mathcal{M} \xRightarrow{\alpha\otimes 1} \mathcal{A}\mathcal{M} \xrightarrow{\varphi} \mathcal{M}.$$

The 2-cell of 6.27 decomposes as $\Xi_6 \circ \Xi_5 \circ \Xi_4$ where Ξ_4 is the 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{A}) \xRightarrow{1\otimes\beta''} \mathcal{A} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\varphi'\otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}],$$

Ξ_5 is

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{A}) \xrightarrow{1\otimes c} \mathcal{A}\mathcal{A} \xRightarrow{\beta''} [\mathcal{M}, \mathcal{M}]$$

and Ξ_6 is

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xRightarrow{\alpha} \mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}].$$

The 2-cell Ξ_1 has image by Rn the 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{A}) \xrightarrow{Rn(A')} [\mathcal{M}, \mathcal{A}((\mathcal{A}\mathcal{A})\mathcal{M})] \xRightarrow{[1,1\otimes\beta]} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\varphi]} [\mathcal{M}, \mathcal{M}]$$

The 2-cell

$$\Xi_7 = \ \mathcal{A}(\mathcal{A}\mathcal{A}) \xrightarrow{Rn(A')} [\mathcal{M}, \mathcal{A}((\mathcal{A}\mathcal{A})\mathcal{M})] \xRightarrow{[1,1\otimes\beta]} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\varphi]} [\mathcal{M}, \mathcal{M}]$$

has a strict domain and according to Lemma [Sch08]-19.6its image by Rn is

$$\Xi_8 = \ \mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xrightarrow{[\beta'',1]} [\mathcal{A}\mathcal{A}, [\mathcal{M}, \mathcal{M}]].$$

The 2-cell

$$\Xi_9 = \ \mathcal{A}(\mathcal{A}\mathcal{A}) \xRightarrow{1\otimes\beta''} \mathcal{A} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\varphi'\otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

has also a strict domain and its image by Rn is Ξ_8, therefore $\Xi_7 = \Xi_9$ and $Rn(\Xi_1) = \Xi_7 * A' = \Xi_9 * A' = \Xi_4$.

The image by Rn of Ξ_2 is

$$(\mathcal{A}\mathcal{A})\mathcal{A} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{A}) \xrightarrow{1\otimes c} \mathcal{A}\mathcal{A} \xrightarrow{Rn(\beta)} [\mathcal{M},\mathcal{M}]$$

which is Ξ_5.

Eventually the image by Rn of Ξ_3 is trivially Ξ_6. ∎

To prove 6.29, we shall use the following lemma.

Lemma 7.43 *For any objects $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, any 2-cell $\mathcal{C} \overset{\tau}{\Longrightarrow} [\mathcal{D},\mathcal{A}]$ of SPC and any $c \in \mathcal{C}$, the 2-cells in SPC*

$$[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{D},-]} [[\mathcal{D},\mathcal{A}],[\mathcal{D},\mathcal{B}]] \overset{[\tau,1]}{\Longrightarrow} [\mathcal{C},[\mathcal{D},\mathcal{B}]] \xrightarrow{ev_c} [\mathcal{D},\mathcal{B}]$$

and

$$[\mathcal{A},\mathcal{B}] \xrightarrow{[ev_c,1]} [[\mathcal{C},\mathcal{A}],\mathcal{B}] \overset{[\tau^*,1]}{\Longrightarrow} [\mathcal{D},\mathcal{B}]$$

are equal.

PROOF:According to Lemma [Sch08]-11.9, The first of the 2-cell rewrites

1. $[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{D},-]} [[\mathcal{D},\mathcal{A}],[\mathcal{D},\mathcal{B}]] \overset{[\tau,1]}{\Longrightarrow} [\mathcal{C},[\mathcal{D},\mathcal{B}]] \xrightarrow{D} [\mathcal{D},[\mathcal{C},\mathcal{B}]] \xrightarrow{[1,ev_c]} [\mathcal{D},\mathcal{B}]$
according to Lemma [Sch08]-10.9,
2. $[\mathcal{A},\mathcal{B}] \xrightarrow{[\mathcal{C},-]} [[\mathcal{C},\mathcal{A}],[\mathcal{C},\mathcal{B}]] \xrightarrow{[\tau^*,1]} [\mathcal{D},[\mathcal{C},\mathcal{B}]]$

and this last arrow has dual

$$\mathcal{D} \overset{\tau^*}{\Longrightarrow} [\mathcal{C},\mathcal{A}] \xrightarrow{[-,\mathcal{B}]} [[\mathcal{A},\mathcal{B}],[\mathcal{C},\mathcal{B}]] \xrightarrow{[1,ev_c]} [\mathcal{A},\mathcal{B}],\mathcal{B}].$$

One the other hand the dual of the second 2-cell of the lemma rewrites

1. $\mathcal{D} \overset{\tau^*}{\Longrightarrow} [\mathcal{C},\mathcal{A}] \xrightarrow{[ev_c,1]^*} [[\mathcal{A},\mathcal{B}],\mathcal{B}]$
2. $\mathcal{D} \overset{\tau^*}{\Longrightarrow} [\mathcal{C},\mathcal{A}] \xrightarrow{ev_c} \mathcal{A} \xrightarrow{q} [[\mathcal{A},\mathcal{B}],\mathcal{B}]$
3. $\mathcal{D} \overset{\tau^*}{\Longrightarrow} [\mathcal{C},\mathcal{A}] \xrightarrow{[-,\mathcal{B}]} [[\mathcal{A},\mathcal{B}],[\mathcal{C},\mathcal{B}]] \xrightarrow{[1,ev_c]} [[\mathcal{A},\mathcal{B}],\mathcal{B}]$

where in the above derivation arrows 2. and 3. are equal by Corollary [Sch08]-11.4. ∎

7.44 *Proof of 6.29.*

PROOF:The 2-cell from 6.29 decomposes as $\zeta_2 \circ \zeta_1$ where $\zeta_1 =$

$$\mathcal{A} \xrightarrow{R'} \mathcal{A}\otimes\mathcal{I} \overset{1\otimes\gamma'}{\Longrightarrow} \mathcal{A}\otimes[\mathcal{M},\mathcal{M}] \xrightarrow{\varphi'\otimes 1} [\mathcal{M},\mathcal{M}]\otimes[\mathcal{M},\mathcal{M}] \xrightarrow{c} [\mathcal{M},\mathcal{M}]$$

and $\zeta_2 =$

$$\mathcal{A} \xrightarrow{R'} \mathcal{A}\otimes\mathcal{I} \xrightarrow{1\otimes u} \mathcal{A}\otimes\mathcal{A} \overset{\beta''}{\Longrightarrow} [\mathcal{M},\mathcal{M}].$$

We show below that the image by Rn of the 2-cell Ξ_2 is ζ_1 whereas the image by Rn of Ξ_3 is ζ_2.

The image by Rn of the 2-cell Ξ_2 rewrites

1. $\mathcal{A} \xrightarrow{\eta} [\mathcal{A}, \mathcal{A}\mathcal{M}] \xRightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{A} \otimes \mathcal{M}] \xrightarrow{[1,\varphi]} [\mathcal{M}, \mathcal{M}]$

2. $\mathcal{A} \xrightarrow{\eta} [\mathcal{M}, \mathcal{A}\mathcal{M}] \xrightarrow{[1,\varphi]} [\mathcal{M}, \mathcal{M}] \xRightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{M}]$

3. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xRightarrow{[\gamma,1]} [\mathcal{M}, \mathcal{M}]$

According to Lemma 7.43 and since the 2-cell γ is $ev_\star * (\gamma')$, the 2-cell 3. above is

$$\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xRightarrow{[\gamma',1]} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{M}].$$

This last 2-cell is actually ζ_1. To check this use Lemma 7.21 and the fact that the image by Rn of the arrow

$$\mathcal{A} \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{\varphi' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

is

$$\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[\mathcal{M},-]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]].$$

The image by Rn of the 2-cell Ξ_3 is $\mathcal{A} \xrightarrow{R'} \mathcal{A} \otimes \mathcal{I} \xrightarrow{1 \otimes u} \mathcal{A} \otimes \mathcal{A} \xRightarrow{Rn(\beta)} [\mathcal{M}, \mathcal{M}]$ which is ζ_2.

∎

7.45 *Proof of 6.31.*

PROOF:The 2-cell Ξ_2 rewrites

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{M} \arrow[r, "\varphi"] & \mathcal{M} \arrow[rr, dash, "id"] \arrow[d, "\varphi'^*"'] \arrow[drr, phantom, "\Downarrow \gamma''"] & & \mathcal{M} \\
& {[\mathcal{A}, \mathcal{M}]} \arrow[rr, "{[u,1]}"'] & & {[\mathcal{I}, \mathcal{M}]} \arrow[u, "ev_\star"']
\end{tikzcd}$$

which image by Rn is the 2-cell

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi'"] & {[\mathcal{M}, \mathcal{M}]} \arrow[rr, dash, "id"] \arrow[d, "{[1,\varphi'^*]}"'] \arrow[drr, phantom, "\Downarrow {[1,\gamma'']}"] & & {[\mathcal{M}, \mathcal{M}]} \\
& {[\mathcal{M}, [\mathcal{A}, \mathcal{M}]]} \arrow[rr, "{[1,[u,1]]}"'] & & {[\mathcal{M}, [\mathcal{I}, \mathcal{M}]].} \arrow[u, "{[1,ev_\star]}"']
\end{tikzcd}$$

On the other hand the 2-cell

$$\mathcal{A} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{A} \xrightarrow{1 \otimes \varphi'} \mathcal{I} \otimes [\mathcal{M}, \mathcal{M}] \xRightarrow{\gamma' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$$

rewrites

1. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{L'} \mathcal{I} \otimes [\mathcal{M}, \mathcal{M}] \xRightarrow{\gamma' \otimes 1} [\mathcal{M}, \mathcal{M}] \otimes [\mathcal{M}, \mathcal{M}] \xrightarrow{c} [\mathcal{M}, \mathcal{M}]$

2. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xrightarrow{[-,\mathcal{M}]} [[\mathcal{M}, \mathcal{M}], [\mathcal{M}, \mathcal{M}]] \xRightarrow{[\gamma',1]} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{M}]$

3. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xRightarrow{[1,\gamma'^*]} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]] \xrightarrow{D} [\mathcal{I}, [\mathcal{M}, \mathcal{M}]] \xrightarrow{ev_\star} [\mathcal{M}, \mathcal{M}]$

4. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xRightarrow{[1,\gamma'^*]} [\mathcal{M}, [\mathcal{I}, \mathcal{M}]] \xrightarrow{[1,ev_\star]} [\mathcal{M}, \mathcal{M}]$

5. $\mathcal{A} \xrightarrow{\varphi'} [\mathcal{M}, \mathcal{M}] \xRightarrow{[1,\gamma'']} [\mathcal{M}, \mathcal{M}].$

In the above derivation the equality between arrows hold for the following reasons:
- 1. and 2. by Lemma 7.8,
- 2. and 3. by Lemma [Sch08]-10.8(which can be improved to take 2-cells into account),
- 3. and 4. by Lemma [Sch08]-11.9.

By definition of the 2-cell θ, the 2-cell Ξ_3 has image by Rn an identity 2-cell. Eventually it is rather straightforward that 2-cell Ξ_4 has image by Rn

$$\mathcal{A} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{A} \xrightarrow{u\otimes 1} \mathcal{A} \otimes \mathcal{A} \xRightarrow{\beta''} [\mathcal{M}, \mathcal{M}]$$

since the 2-cell β'' is $Rn(\beta)$. ∎

7.46 *PROOF of 6.32*

PROOF:The 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{M} \xrightarrow{1\otimes H} \mathcal{A}\mathcal{A}\mathcal{N} \xRightarrow{\beta} \mathcal{N}$$

has a strict domain

$$\xrightarrow{1\otimes H} \; \xrightarrow{A'} \; \xrightarrow{1\otimes \psi} \; \xrightarrow{\psi}$$

and has image by Rn

$$\mathcal{A}\mathcal{A} \xRightarrow{\beta''} [\mathcal{N}, \mathcal{N}] \xrightarrow{[H,1]} [\mathcal{M}, \mathcal{N}]$$

which is image by Rn is Ξ_1. The 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{M} \xrightarrow{c\otimes 1} \mathcal{A}\mathcal{M} \xRightarrow{\delta} \mathcal{N}$$

has image by Rn the 2-cell

$$\mathcal{A}\mathcal{A} \xrightarrow{c} \mathcal{A} \xRightarrow{\delta'} [\mathcal{M}, \mathcal{N}]$$

which image by Rn is Ξ_2. ∎

7.47 *Proof of 6.33.*

PROOF:According to Lemma 7.21, the 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{M} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{M}) \xRightarrow{1\otimes \delta} \mathcal{A}\mathcal{N} \xrightarrow{\psi} \mathcal{N}$$

has image by $Rn \circ Rn$ the 2-cell Ξ_4. According to Lemma 7.21, the 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{M} \xrightarrow{A'} \mathcal{A}(\mathcal{A}\mathcal{M}) \xrightarrow{1\otimes \varphi} \mathcal{A}\mathcal{M} \xRightarrow{\delta} \mathcal{N}$$

has image by $Rn \circ Rn$ the 2-cell Ξ_6. Eventually the 2-cell

$$(\mathcal{A}\mathcal{A})\mathcal{M} \xRightarrow{\beta} \mathcal{M} \xrightarrow{\sigma} \mathcal{N}$$

has image by $Rn \circ Rn$ the 2-cell Ξ_7. ∎

7.48 *Proof of 6.34.*

PROOF:The 2-cell
$$u^2{}_H : \; H \to [1,H] \circ v : \mathcal{I} \to [\mathcal{M},\mathcal{N}]$$
has image by $ev_\star$ an identity 2-cell and the 2-cell

$$\mathcal{I} \xRightarrow{\gamma'} [\mathcal{M},\mathcal{M}] \xrightarrow{[1,H]} [\mathcal{M},\mathcal{N}]$$

has image by $ev_\star$ the 2-cell
$$\mathcal{M} \xRightarrow{\gamma} \mathcal{M} \xrightarrow{H} \mathcal{N}$$

∎

7.49 *Proof of 6.35.*

PROOF:The arrow $H : \mathcal{I} \to [\mathcal{M},\mathcal{N}]$ is strict has image by the functor $ev_\star$ the arrow $H : \mathcal{M} \to \mathcal{N}$. The 2-cell
$$\mathcal{I} \xRightarrow{\gamma'} [\mathcal{N},\mathcal{N}] \xrightarrow{[H,1]} [\mathcal{M},\mathcal{N}]$$
has image by $ev_\star$ the 2-cell
$$\mathcal{M} \xrightarrow{H} \mathcal{N} \xRightarrow{\gamma} \mathcal{N}.$$
The 2-cell
$$\mathcal{I} \xrightarrow{u} \mathcal{A} \xRightarrow{\delta'} [\mathcal{M},\mathcal{N}]$$
has image by $ev_\star = SMC(1, ev_\star) \circ D$ the 2-cell

$$\mathcal{M} \xRightarrow{Rn(\delta)^*} [\mathcal{A},\mathcal{N}] \xrightarrow{[u,1]} [\mathcal{I},\mathcal{N}] \xrightarrow{ev_\star} \mathcal{N}$$

which is according to Lemma 7.8

$$\mathcal{M} \xrightarrow{L'} \mathcal{I} \otimes \mathcal{M} \xrightarrow{u \otimes 1} \mathcal{A} \otimes \mathcal{M} \xRightarrow{\delta} \mathcal{M}$$

∎

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